11/7/12

Combinatorial Games

Ex.: Game 1

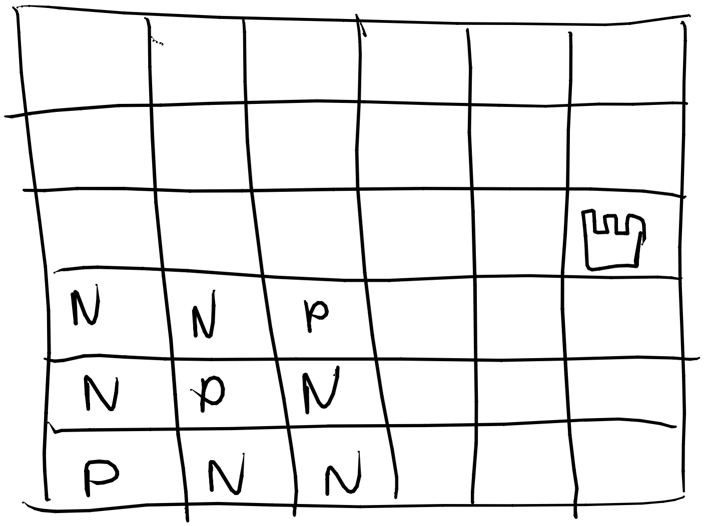
Two players A(lice) and B(ob) take turns.
Start with n chips and A, B play alternately,
taking 1, 2, 3 or 4 chips until pile is empty.
Last player to move wins.

Two sorts of position: N position, winning position
Next player wins

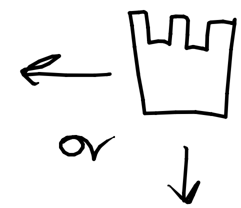
losing $\rightarrow$ P position, Previous player wins.

Game 2

(a)



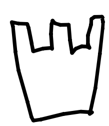
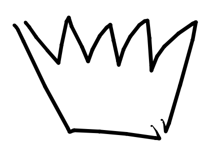
Chessboard



Players move the "rook" any number to left or down.

Strategy – if not on diagonal, move to diagonal.

– 2 pile NIM.

(b) Replace  by  Queen. Wythoff's NIM

– more complicated.

3)

Geography

W is a set of words. A & B alternately remove words from W.

Rule: ω_{i+1} must start with last letter of ω_i

USA	ALBANIA	AZERBAIJAN	NEPAL	
A	B	A	B	- - - -

Solved using <u>Matchings</u>

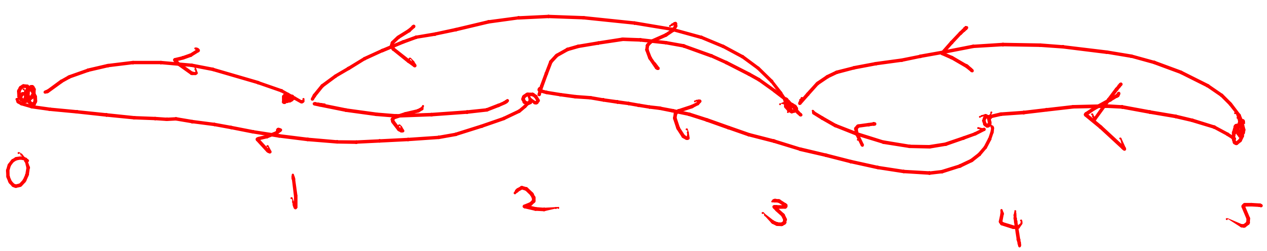
Abstraction

Represent the game by a digraph $D = (X, A)$

$X = \{\text{positions}\}$

$A = \{ (p, q) \}$ such there is move from $p \to q$

Example Pile of 5 chips, can remove 1 or 2.



We place a "marker" on D to mark current position

Player moves the marker along an edge, in the proper direction.

Game ends when the marker is at x, where

$N^+(x) = \emptyset$ — x is a <u>sink</u>

out-nbrs of x = $\{ y : (x,y) \in A \}$

We will assume

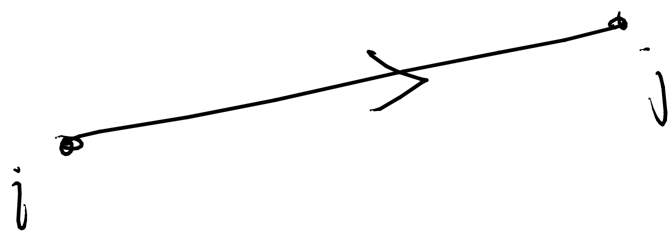
(i) X is finite

(ii) D is a DAG — no directed cycles.

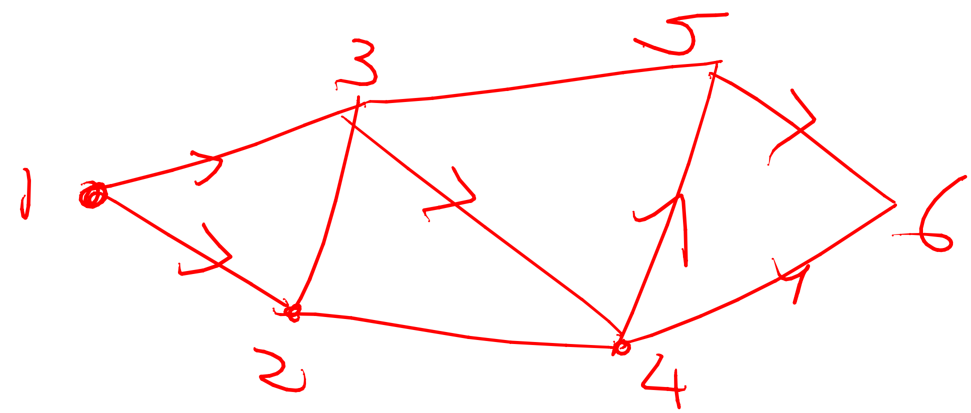
Topological Ordering

Suppose $|X| = n$

Topological ordering of D is an assignment of $1, 2, \ldots, n$ to X so that



$i < j$

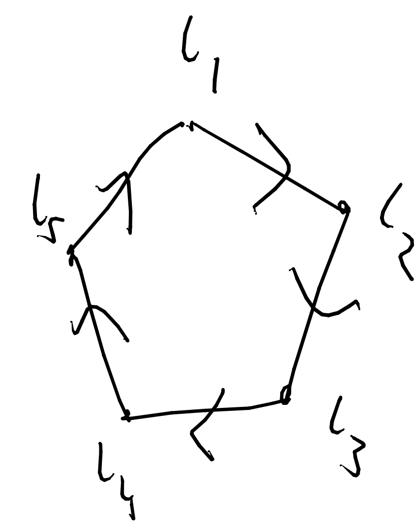


Claim

A finite digraph is a DAG iff it has a topological ordering.

Proof

① if $\exists$ topological ordering – no cycles



$l_1 < l_2 < \dots < l_5$

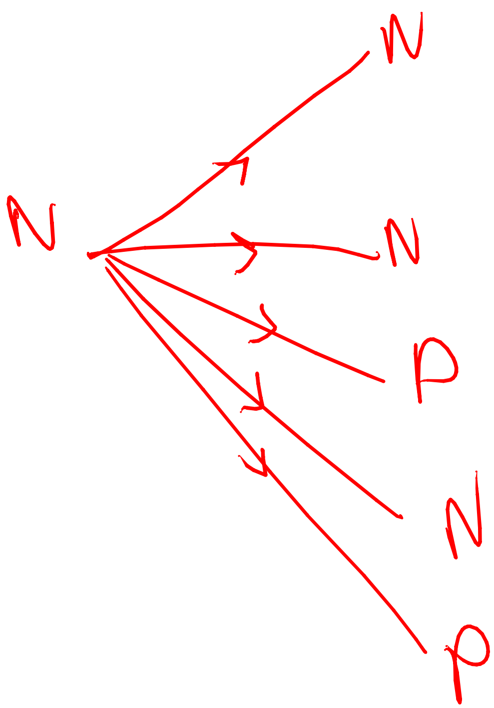
⑪ If D is a DAG, then D has a sink – one end of a longest path.

Numbe this n and use induction

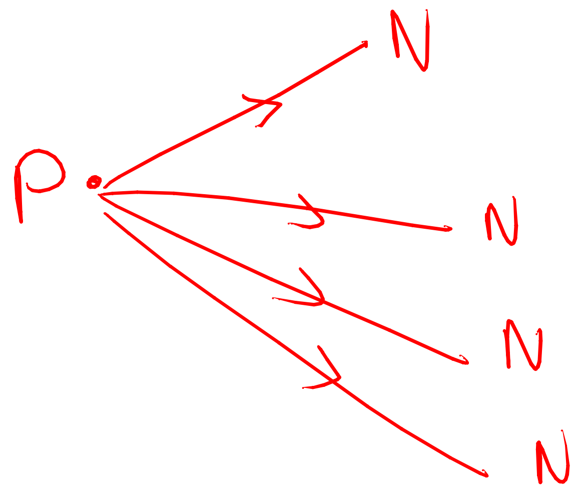
N positions and P positions

winning

losing



N sees at least one P position



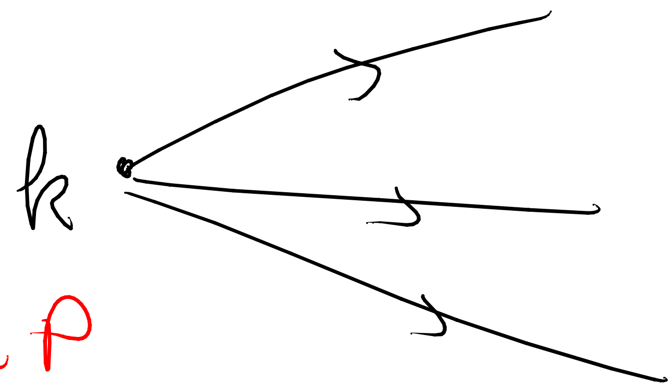
P only "sees" N positions

Labelling for N & P positions:

(i) Topologically order the vertices $1, 2, \ldots, n$

(ii) Position n will be a P-position.

Suppose we have labelled $k+1, \ldots, n$



Nbrs have been labelled.

So label of k is determined.

Sets N & P are unique.

Sums of Games

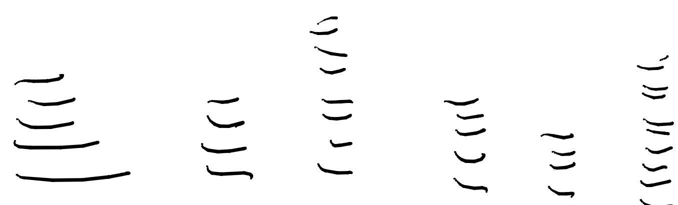
Suppose that we have games $G_1, G_2, \ldots, G_p$ with digraph $D_i = (X_i, A_i)$, $i = 1, 2, \ldots, p$.

We define

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_p$$

To play G, choose i and make a move in G_i.

Simplest example NIM



Move: choose non-empty pile, remove non-zero number of chips

In terms of digraph:

$D = (X, A)$ where

$$X = X_1 \times X_2 \times \cdots \times X_p$$

$$A = \left[(x_1, x_2, \ldots, x_p), \; (x'_1, x'_2, \ldots, x'_p) \right]$$

where $\exists\, j :$ $\quad x_i = x'_i \qquad i \neq j$

$$x'_j \in N^+(x_j)$$

Need a mechanism for finding winning strategy for sums of games.

(SPRAGUE)- GRUNDY NUMBERS

This is a numbering of the positions that enables us to solve sums of games.

Knowing GRUNDY numbers tells you how to play the game

Knowing GRUNDY for G_1, G_2 means I can find GRUNDY for $G_1 \oplus G_2$

If $S = \{x_1, x_2, \dots, x_k\} \subseteq \{0, 1, 2, \dots\}$

$$\text{mex}(S) = \min\{x \notin S\}$$

minimum
excluded

Numbering of D

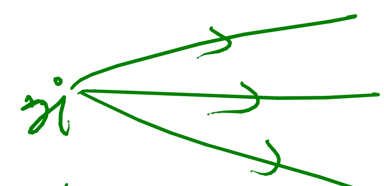
$$\text{mex}(\{0, 1, 3, 5\}) = 2$$

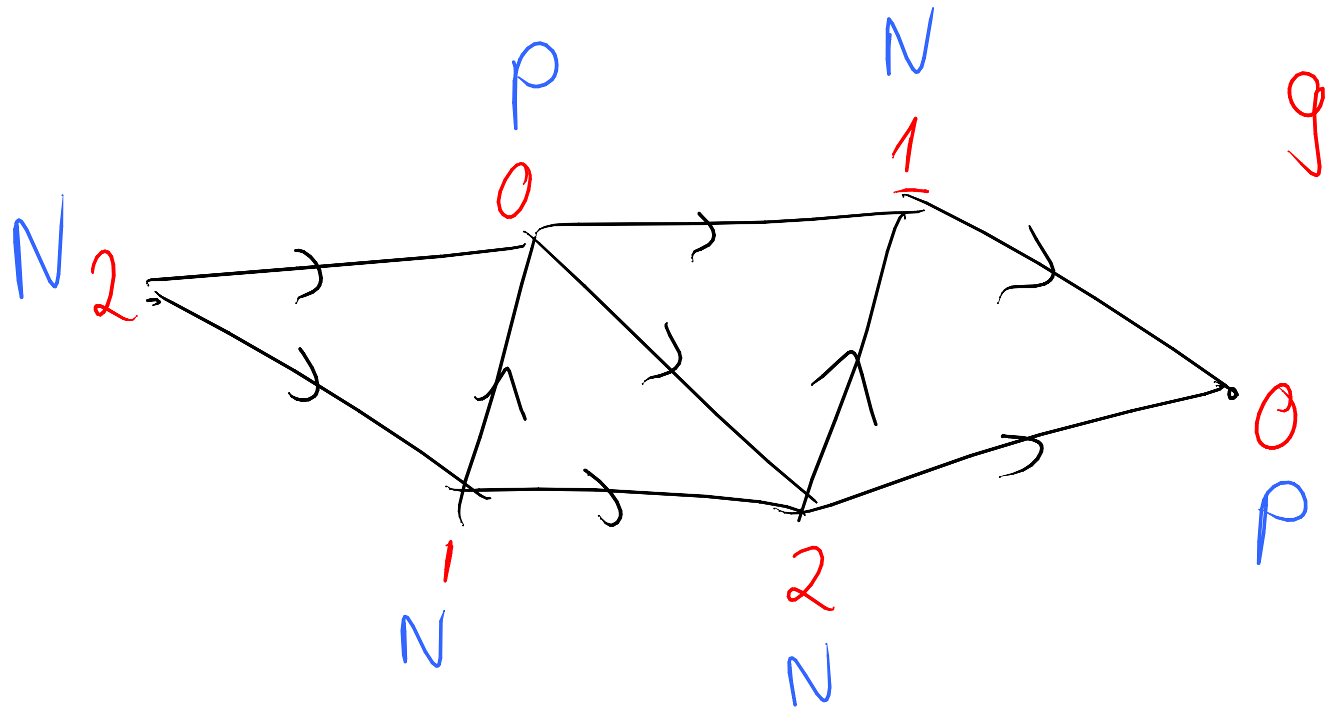
$$\text{mex}(\{1, 2, 3, 4 \dots\}) = 0$$

n

$g(n) = 0$

$$g(x) = \text{mex}(\{g(y) : y \in N^+(x)\})$$





One property we have is that $g(x) = 0$

$\Longleftrightarrow x \in P$