5\11\12

Proof of Dilworth's theorem.

By induction on the size of the poset P.

Let μ = maximum size of an anti-chain in P.
We show that P can be partitioned into μ chains.

Let $C = x_1 < x_2 < \cdots < x_p$ be a longest chain in P.

Case 1

Every antichain in $P \setminus C$ has $\leq \mu - 1$ elements

By induction, $P \setminus C$ can be covered by $\mu - 1$ chains $C_2, C_3, \ldots, C_\mu$

and then $C, C_2, \ldots, C_\mu$ cover P

Case 2 $\exists$ an anti-chain $A = \{a_1, a_2, \ldots, a_\mu\} \subseteq P \setminus C$

(i) $P = P^- \cup P^+$: every element is comparable to something in A, else A is not the largest anti-chain.

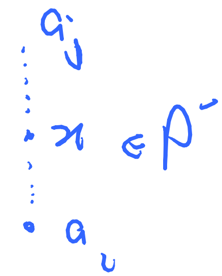
(ii) $A = P^- \cap P^+$: If not, and $x \notin A$ & $x \in P^- \cap P^+$ then $a_i < x < a_j$ for some i, j.

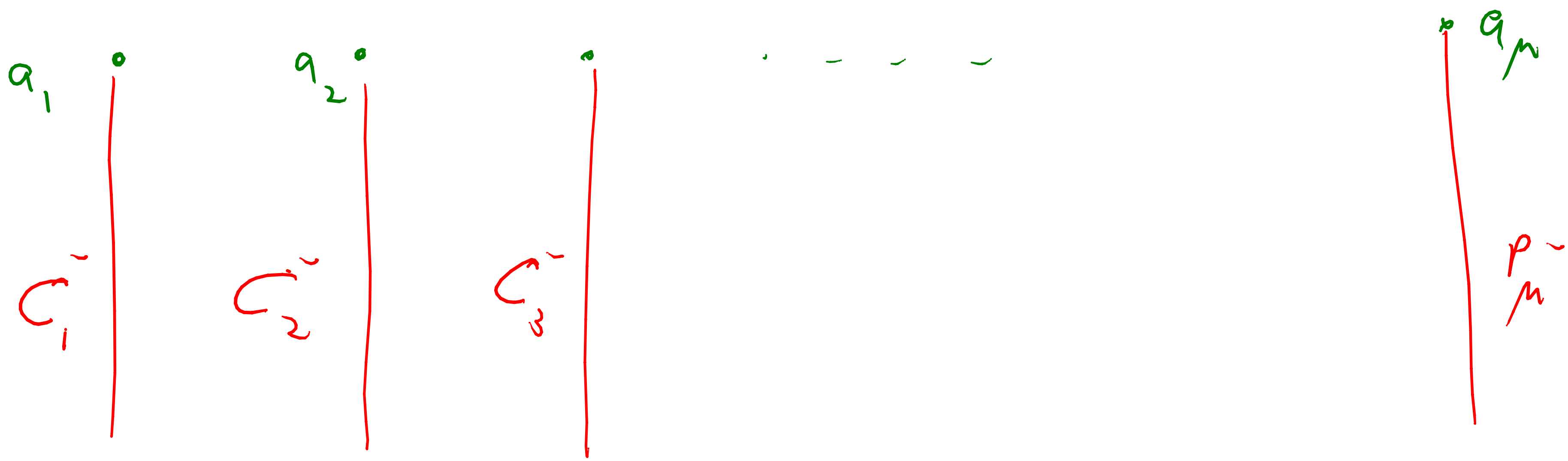
(iii) $x_p \notin P^-$: $x_p \notin A$, if $x_p \in P^-$ then $x_p < a_i$ and C can be made bigger.

We can use induction on P^-. We can partition P^- into μ chains, $C_1^-, C_2^-, \ldots C_\mu^-$.

$$P^- = \{x \in P : \exists i \text{ s.t. } x \le a_i\} \text{ and } P^+ = \{x \in P : \exists i \text{ s.t. } x \ge a_i\}$$

Claim: a_i is largest element in $\tilde{C}_i$. If not



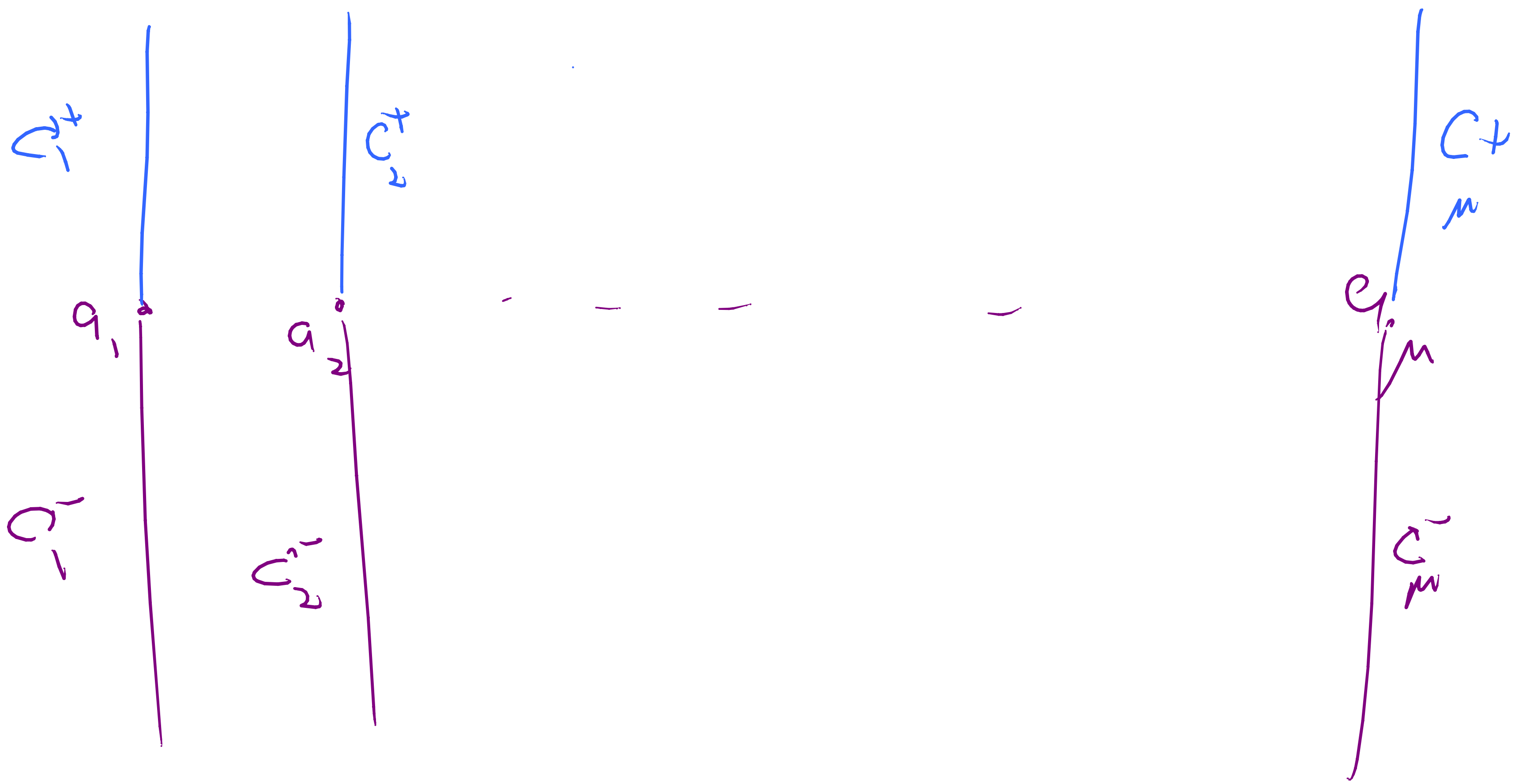


$\tilde{C}_1 \cup \tilde{C}_2 \cup \dots \cup \tilde{C}_n = \tilde{P}$.

Assume $a_i \in \tilde{C}_i$, $i = 1, 2, \dots, n$

If $i \neq j$ then a_i & a_j are in different chains.

Do the same for P^+



$C_i = C_i^- \cup C_i^+$ is a chain decomposition.

Proof of Hall's Theorem for matchings in bipartite graphs.

$G = (A \cup B, E)$ and suppose that Hall's condition holds.

Define poset $P = A \cup B$

$a < b$ if $a \in A$
$b \in B$
$(a,b) \in E$

Suppose largest anti-chain in P is

$$X = \{a_1, a_2, \dots, a_h, b_1, b_2, \dots, b_k\}, \quad s = h+k$$

Because X is an anti-chain

$$N(\{a_1, a_2, \dots, a_h\}) \subseteq B \setminus \{b_1, b_2, \dots, b_k\}$$

From Hall's condition we know that

$$|B| - k \geq h \quad |B| \geq s.$$

P is the union of s chains:

[a chain is of length at most 2.]

A matching M of size m, $|A| - m$ members of A and

$|B| - m$ members of B

$$m + (|A| - m) + (|B| - m) = s \leq |B|$$

$$\Longrightarrow \quad m \geq |A|.$$