11\30\12

Instead of giving # of orbits we want to compute a sum (2-colors)

$$\sum_{i+j=101} a_{i,j} R^i B^j$$

Putting $R=B=1$ gives # orbits

$a_{i,j}$ = # of colorings with i Red and j Blue

We have a <u>weight</u> or <u>symbol</u> w_c for each color $c \in C$.

If $x \in X$ then

$$W(x) = \prod_{d \in D} w_{x(D)}$$

R	B
R	R

$\leftarrow x$

$$W(x) = R^3 B$$

For a set $S \subseteq X$ we have $W(S) = \sum_{x \in S} W(x)$

The <u>pattern inventory</u> $PI = W(S^*)$

where S^* contains one member of each orbit

Problem: compute PI

Definition has meaning i.e. X, Y in same orbit

$\Longrightarrow$ $W(x) = W(y)$ $\qquad y = g * x$

$$W(y) = \prod_{d \in D} W_{y(d)}$$

$$= \prod_{d \in D} W_{g * x(d)}$$

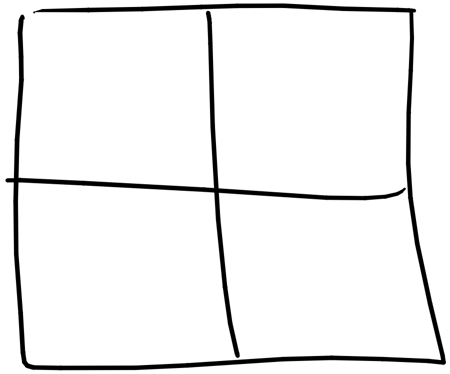
$$= \prod_{d \in D} W_{x(g^{-1}(d))} = \prod_{x \in D} W_{x(d)} = W(x)$$

Let $\Delta = |D|$. Suppose $g \in G$ has k_i cycles of length i.

$$ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}$$

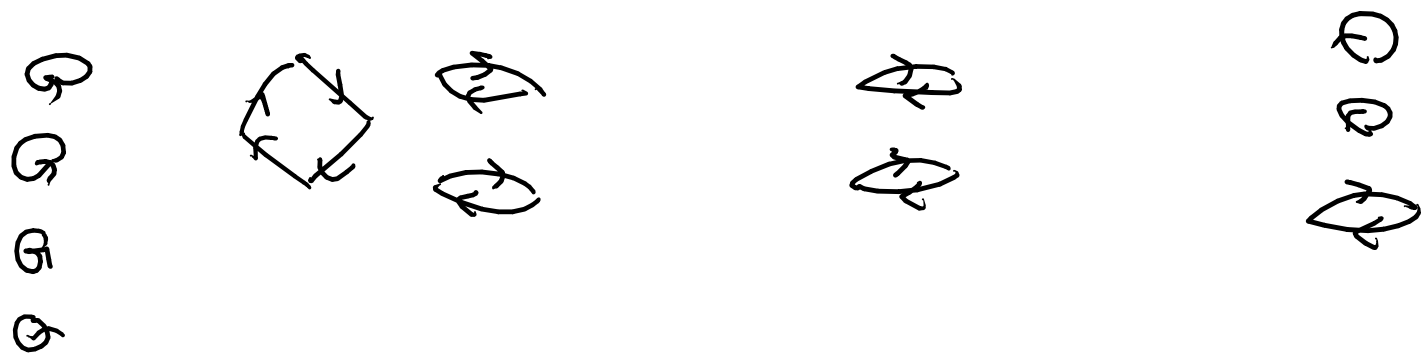
Cycle Index Polynomial

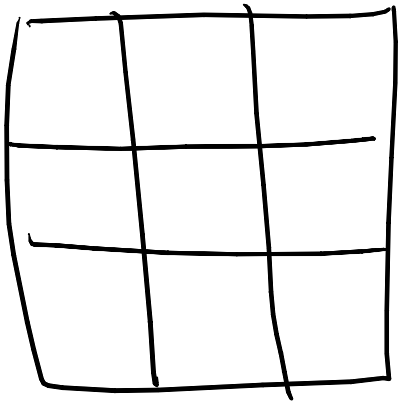
$$\mathcal{G}_G(x_1, \ldots, x_\Delta) = \frac{1}{|G|} \sum_{g \in G} ct(g)$$



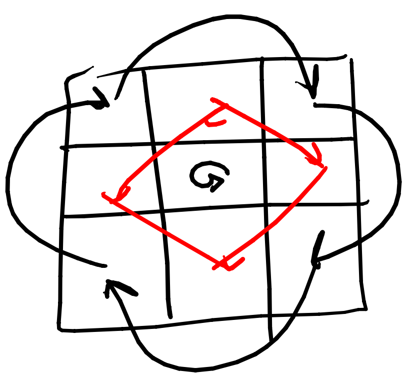
$n=2$

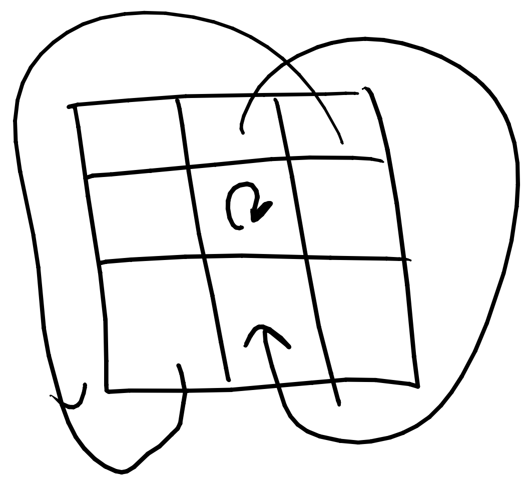
g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^4	x_4	x_2^2	x_4	x_2^2	x_2^2	$x_1^2 x_2$	$x_1^2 x_2$

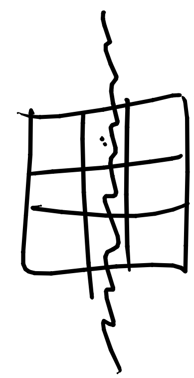




g	e	a	b	c	p	q	r	s
$ct(g)$	x_1^9	$x_1 x_4^2$	$x_1 x_2^4$	$x_1 x_4^2$	$x_1^3 x_2^3$	$x_1^3 x_2^3$	$x_1^3 x_2^3$	$x_1^3 x_2^3$







Theorem

$$PI = C_G\left(R+B+G+\cdots,\ R^2+B^2+G^2+\cdots,\ R^3+B^3+G^3+\cdots,\ \cdots\right)$$

Example 2: $n=2$ $\qquad C_G = \frac{1}{8}\left(x_1^4 + 3x_2^2 + 2x_1^2x_2 + 2x_4\right)$

$$PI = \frac{1}{8}\left((R+B)^4 + 3(R^2+B^2)^2 + 2(R+B)^2(R^2+B^2) + 2(R^4+B^4)\right)$$

$$= R^4 + R^3B + 2R^2B^2 + RB^3 + B^4$$

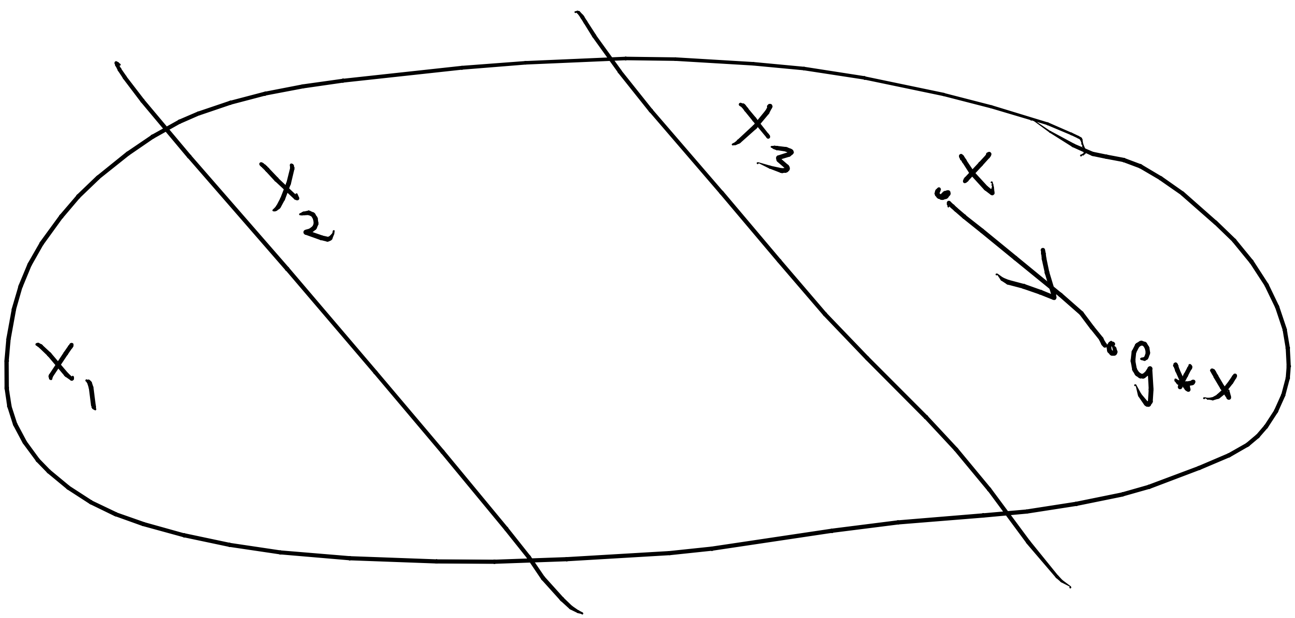
Proof

$$X = X_1 \cup X_2 \cup \cdots \cup X_m$$

be equivalence classes of $\sim$ where

$$x \sim y \quad \text{iff} \quad W(x) = W(y)$$

Each X_i is the union of orbits



X

G also acts on each X_i.

$$PI = \sum_{i=1}^{m} m_i W_i$$

$W_i = W(x),\ x \in X_i$

$m_i = \#$ of orbits $\subseteq X_i$

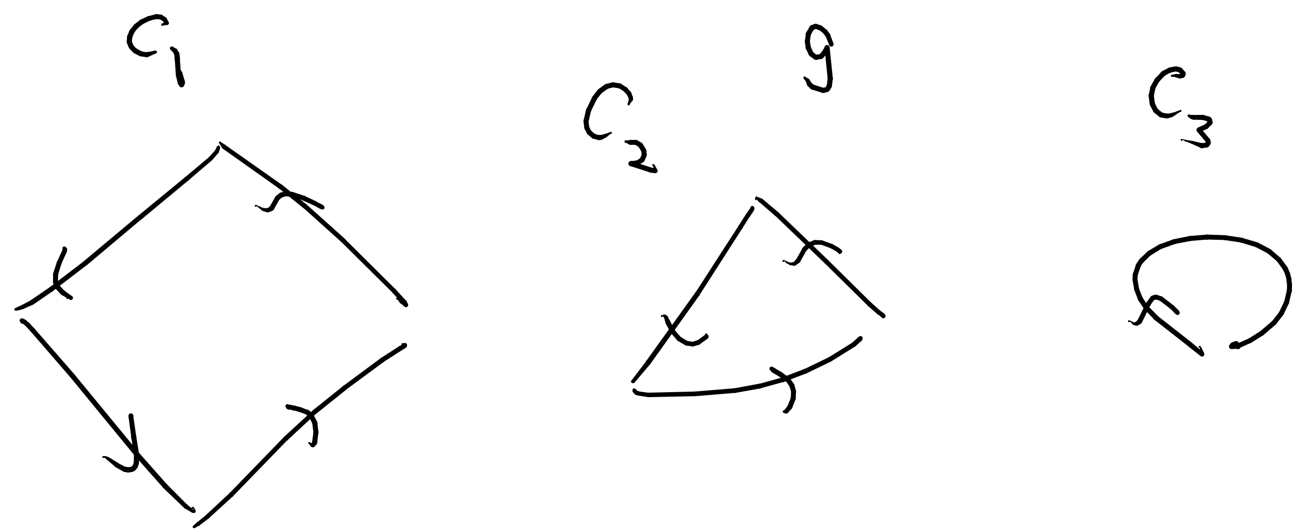
$$= \sum_{i=1}^{m} W_i \left(\frac{1}{|G|} \sum_{g \in G} |Fix(g) \cap X_i| \right)$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{m} |Fix(g) \cap X_i| W_i$$

g has k_i cycles of length i

$$= \frac{1}{|G|} \sum_{g \in G} W(Fix(g)) = \frac{1}{|G|} \sum_{g \in G} (R+B+\dots)^{k_1} (R^2+B^2+\dots)^{k_2} \cdots$$

$$= c_G(\cdots)$$



What patterns arise?

$C_1 \longrightarrow R^4 + B^4 + G^4$

$C_2 \longrightarrow R^3 + B^3 + G^3$

$C_3 \longrightarrow R + B + G$

$$(R^4 + B^4 + G^4)(R^3 + B^3 + G^3) \times$$
$$(R + B + G)$$