2\11\12

Partially Ordered Sets [POSETS]

A poset is a set P plus a binary relation $\leq$ such that

(i) $a \leq a, \ \forall a$ (reflexive)

(ii) $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitive)

(iii) $a \leq b$ and $b \leq a \Rightarrow a = b$ (anti-symmetry

Examples

(i) $P = \{1, 2, \ldots \}$ $\qquad \leq \; = \; \leqslant$ [total order here

(ii) $P = \{1, 2, \ldots \}$ $\qquad a \leq b$ iff a divides b

(iii) $P = \{A_1, A_2, \ldots, A_m\}$ where A_i are sets and

$$\leqslant \; \equiv \; \subseteq$$

a & b are comparable if $a \leq b$ or $a \geq b$

otherwise a, b are incomparable. In a linear order, all pairs are comparable.

$a < b$ if $a \leq b$ and $a \neq b$

A chain is a sequence

$$a_1 < a_2 < a_3 < \cdots < a_5$$

[length $= 5$]

An anti-chain is a set A where

$a, b \in A \Rightarrow a \,\&\, b$ are incomparable.

Sperner family $\equiv$ anti-chain in example 3

$$\emptyset \subseteq \{1,2\} \subseteq \{1,2,5,8\} \subseteq \{1,2,5,8,12\}$$

Chain in example 3

$$\{\{1,2,3\}, \{1,2,4\}, \{2,3,4\}\}$$

is an anti-chain

If C is a chain and A is an anti-chain

then
$$|A \cap C| \leq 1$$

Suppose $x, y \in A \cap C$

$x, y \in C \Rightarrow x < y$ or $y < x$. – contradicts $x, y \in A$

Two theorems relate maximum size of chain (anti-chain) to minimum number of anti-chains (chains) needed to cover P.

A chain cover of P is a set of chains $C_1, C_2, \ldots, C_k$ such that $C_1 \cup C_2 \cup \ldots \cup C_k = P$. Its size is k.

An anti-chain cover is a set of anti-chain $A_1, A_2, \ldots, A_k$ such that $A_1 \cup A_2 \cup \ldots \cup A_k = P$ [size $= k$]

If P can be covered with k chains [anti-chains]

then largest anti-chain [Chain] has size $\leq k$

Pigeon hole principle.

Anti-chain of size $k+1$ would have two [chain] elements in same chain [anti-chain]

<u>Theorem</u> P is a finite poset.

$\nu = \min\{m : P \text{ can be covered with } m \text{ anti-chains}\}$

$\mu =$ max size of a chain.

Then $\mu = \nu$.

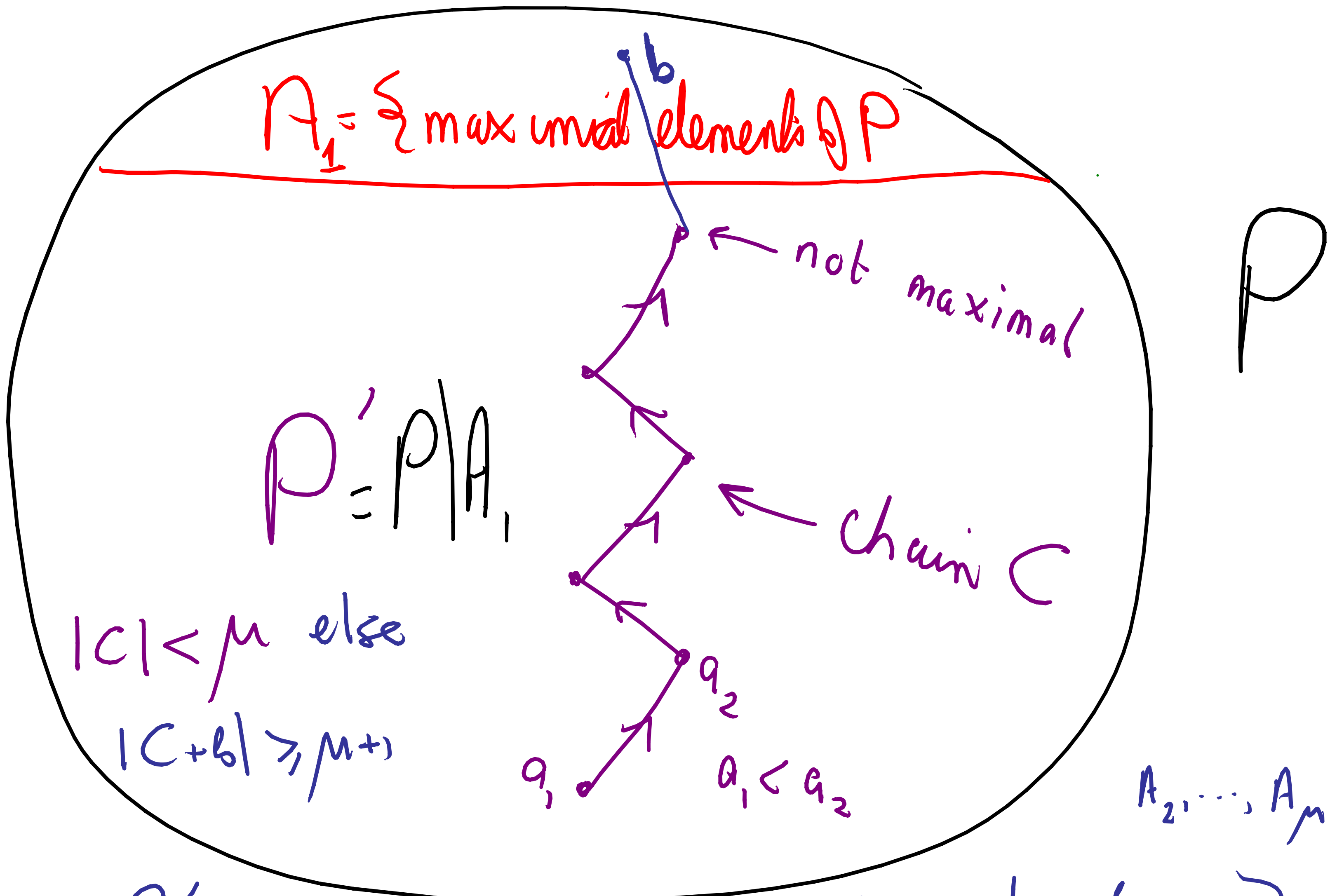
<u>Proof</u>

By induction on μ.

$\mu = 1 \Rightarrow P$ is an anti chain

a is maximal, if $\nexists\, b$ such that $a < b$

μ = maximum chain length



$A_2, \ldots, A_\mu$

By induction, P' can be partitioned into $\mu - 1$ anti-chains.)

$P = \{1, 2, 3, \ldots, 10\}$

$\leq \equiv$ divisibility

$A_1 = \{6, 7, 8, 9, 10\}$

Dilworth's Theorem

P is a finite poset.

$\alpha = \min\{m : P = \underbrace{C_1 \cup C_2 \cup \cdots \cup C_m}_{\text{chains}}\}$

$\beta =$ max size of anti-chain

Then

$$\alpha = \beta$$

Applications

① Erdös, Szekerés : $a_1, a_2, \dots a_{n^2+1}$ contains a monotone subsequence of length $n+1$.

$$P = \{ (i, a_i) : 1 \le i \le n^2+1 \}$$

$$(i, a_i) < (j, a_j) \text{ iff } i < j \text{ and } a_i \le a_j$$

A chain corr. to a monotone increasing subsequence. $(i_1, a_{i_1}) < (i_2, a_{i_2}) < \dots$ $\quad i_1 < i_2$ and $a_{i_1} \le a_{i_2}$ etc

Suppose $\nexists$ a monotone increasing subsequence of length $n+1$

$\implies$ minimum cover by chains need $\geq \left\lceil \frac{n^2+1}{n} \right\rceil = n+1$ chains

$\implies$ $\exists$ an anti-chain of size $n+1$

$(i_1, a_{i_1}), (i_2, a_{i_2}), \ldots (i_{n+1}, a_{i_{n+1}})$

where $i_1 < i_2 < \ldots < i_{n+1}$. So $a_{i_1} > a_{i_2} > \ldots > a_{i_{n+1}}$

N intervals $I_1=[a_1,b_1]$, $I_2=[a_2,b_2]$, . . . , $I_N=[a_N,b_N]$ on the real line $a_i\le b_i$, $1\le i\le N=n^2+1$.

Either $\exists$ $n+1$ disjoint intervals $I_{i_1}, I_{i_2}, \dots I_{i_{n+1}}$

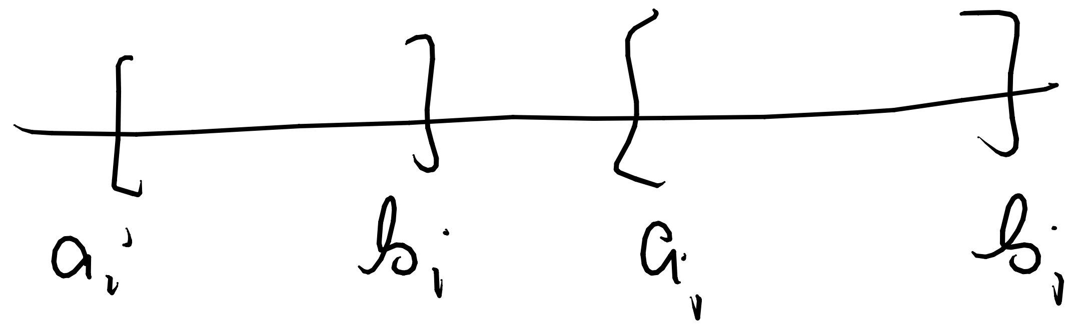
or $\exists$ $n+1$ intervals I_j, $j\in J$ s.t. $\bigcap_{j\in J} I_j\neq\emptyset$

Extrema:

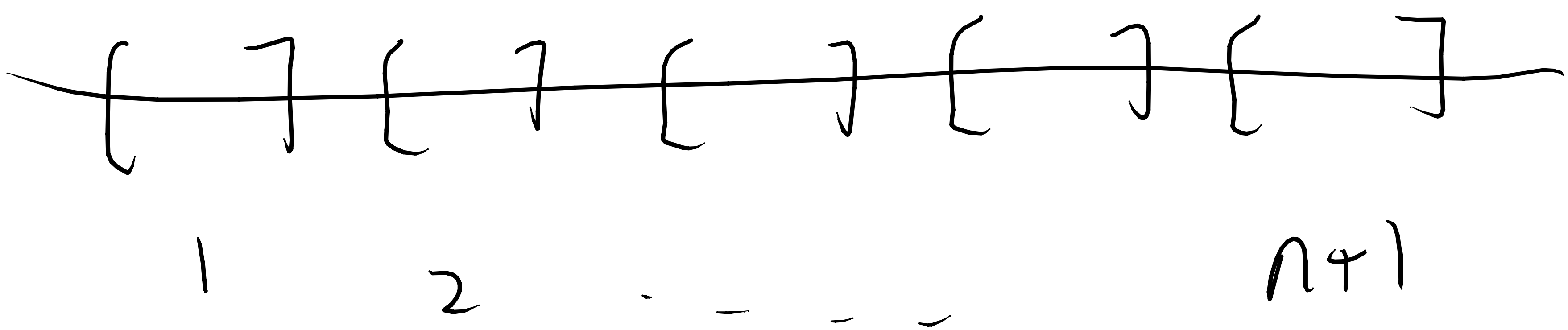
q

$P = \{1, 2, \ldots, N\}$

$i < j$ if



A chain of length $n+1$ looks like



If there is no chain of length $n+1$ then there must be an anti-chain of size

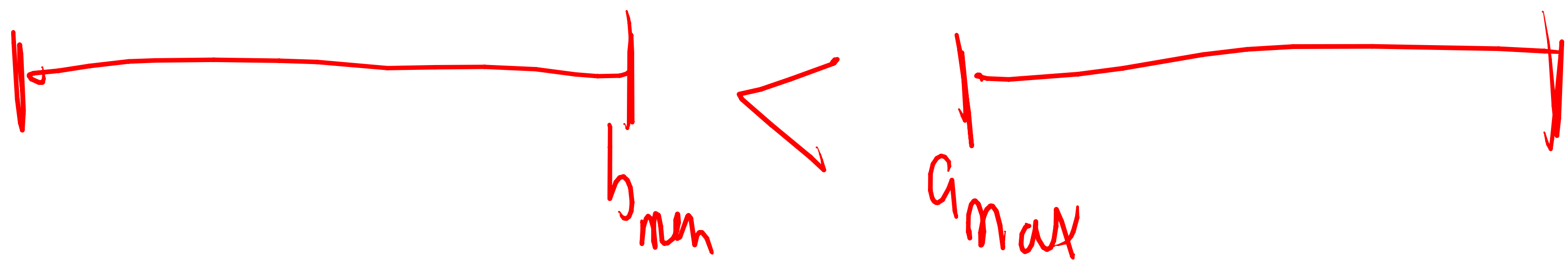
$$\left\lceil \frac{n^2+1}{n} \right\rceil = n+1$$

Anti chain $= I_j, \ j \in J$

$$b_{min} = \text{smallest } b_j$$

$$\text{Claim: } \bigcap_{j \in J} I_j \neq \emptyset \equiv \quad \vee\!/$$

$$a_{max} = \text{largest } a_j$$



b_{min} = smallest b_j

a_{max} = largest a_j

$$[a_{max}, b_{min}] \subseteq I_j \,, \quad j \in J$$