11\28\12

$\mathrm{Fix}(g) = \{x \in X : g * x = x\}$

$$\# \text{ of orbits} = \frac{1}{|G|} \sum_{x \in X} |S_x|$$

$$= \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$$

Burnside's Theorem

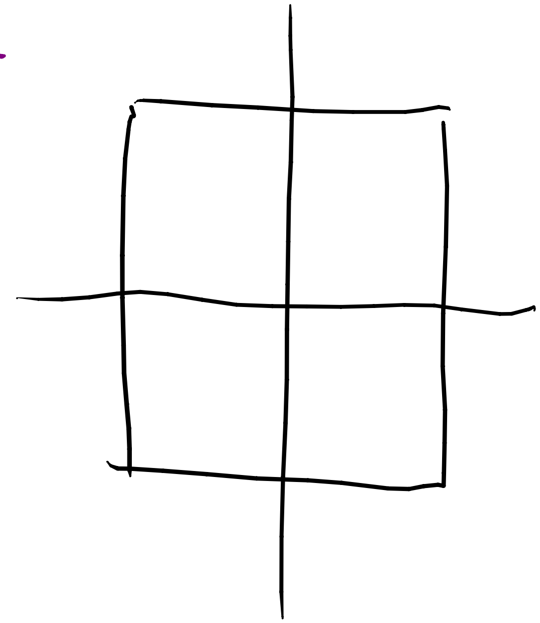
$\Sigma$ = total # of 1's in matrix

$$\begin{array}{cc} & x \\ g & \begin{bmatrix} & & \\ & 1_{g*x=x} & \\ & & \end{bmatrix} \end{array}$$

$\leftarrow$ # 1's = $|\mathrm{Fix}(g)|$

$\uparrow$ # 1's = $|S_x|$

Example 2



$n \times n$ chessboard

$n = 2m$ even

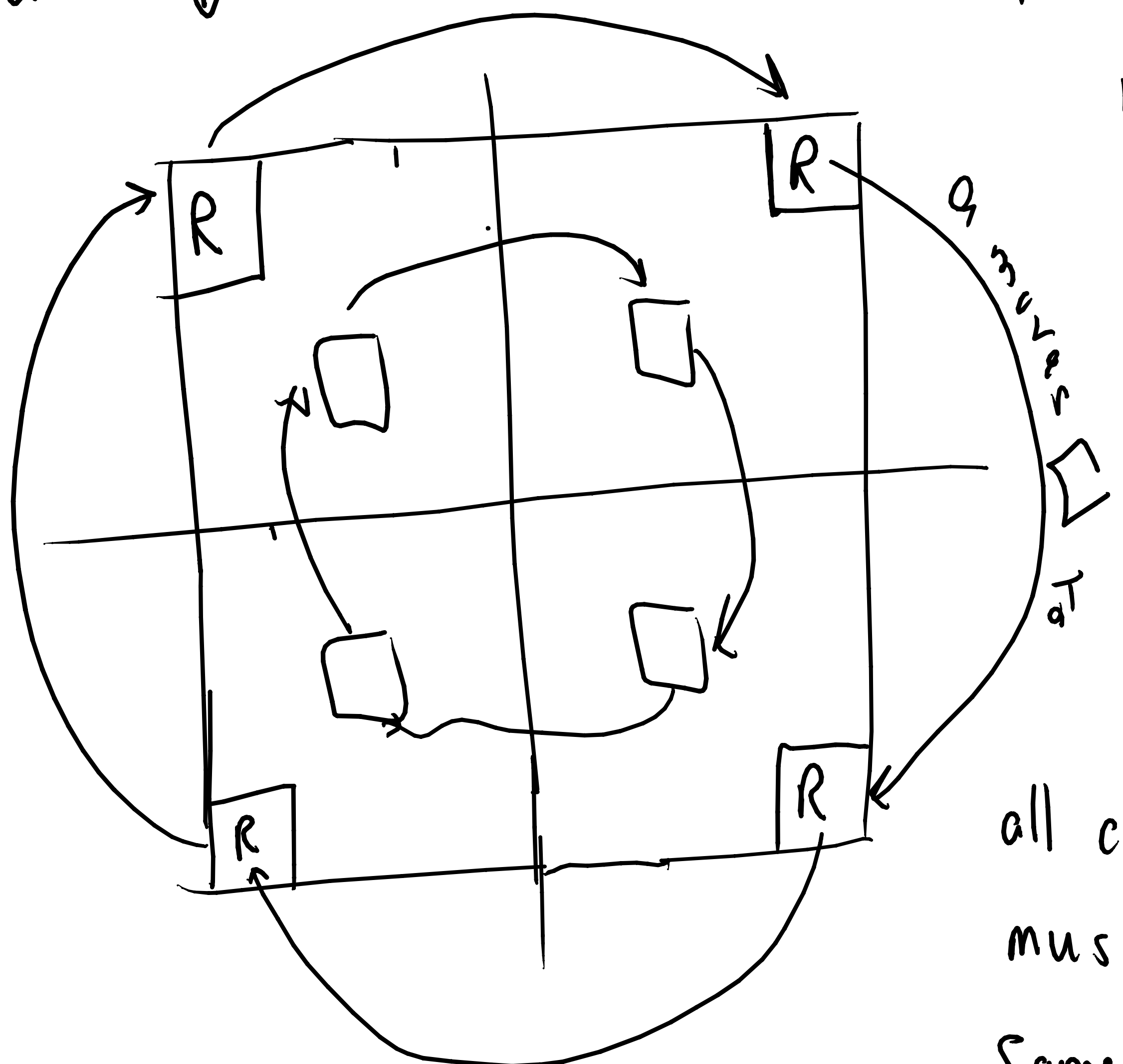
| $g$ | $e$ | $a$ | $b$ | $c$ | $p$ | $q$ | $r$ | $s$ |
|---|---|---|---|---|---|---|---|---|
| $\lvert Fix(g) \rvert$ | $2^{n^2}$ | $2^{m^2}$ | $2^{n^2/2}$ | $2^{m^2}$ | $2^{n^2/2}$ | $2^{n^2/2}$ | $2^{\frac{n(n+1)}{2}}$ | $2^{\frac{n(n+1)}{2}}$ |

$\#G$ colorings $= \frac{1}{8}\left( \quad + \quad \cdots \quad \right)$

$a$: $\curvearrowright 90^\circ$

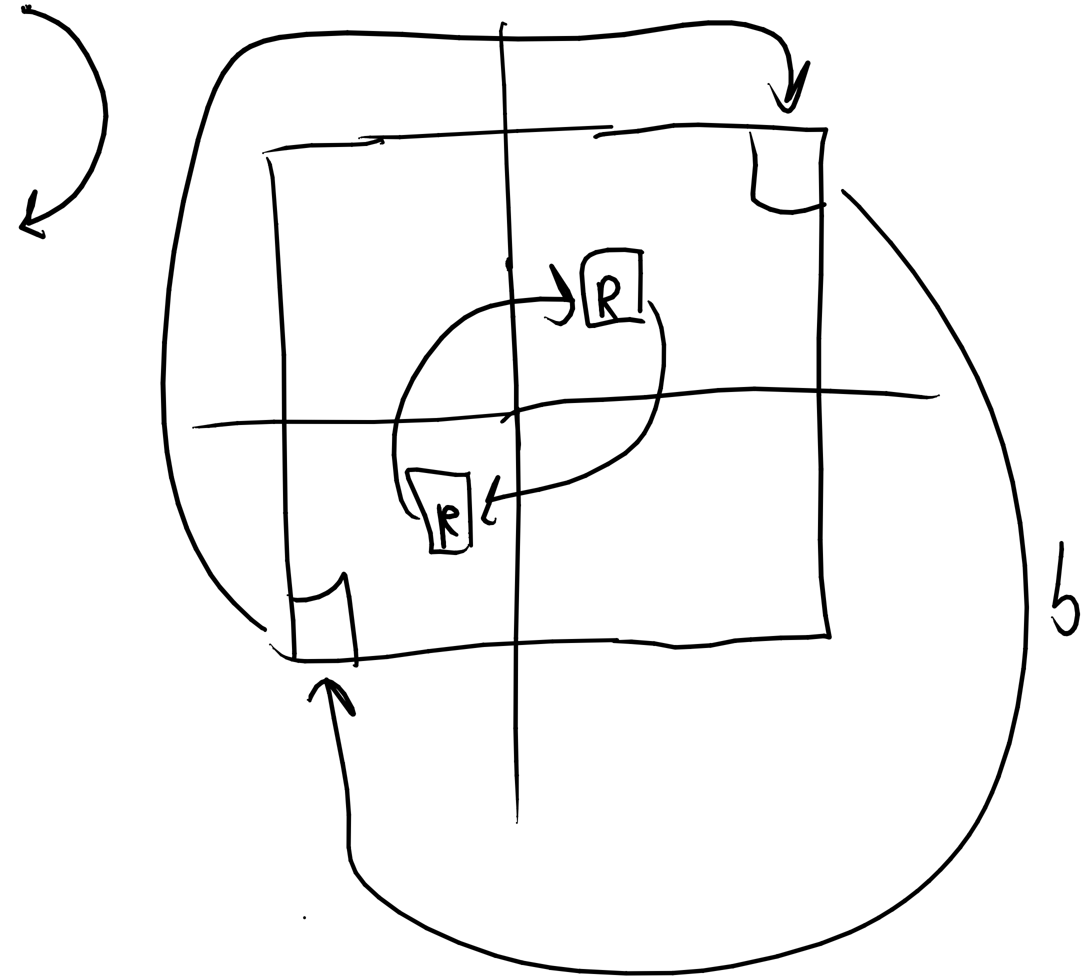
$$|Fix(a)| = 2^{m^2}$$

$n = 2m$

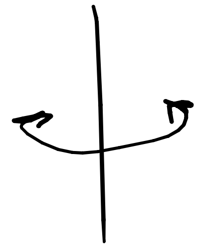


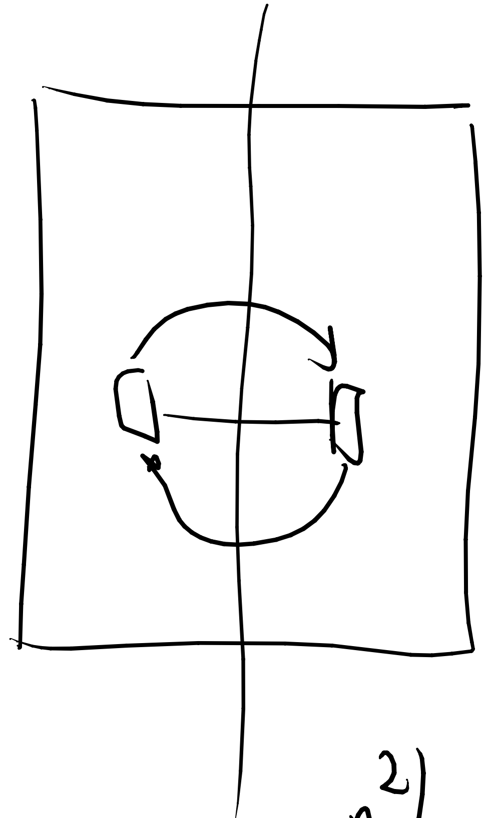

all colors must be the same.

$b =$

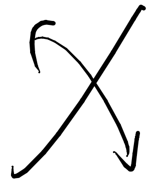


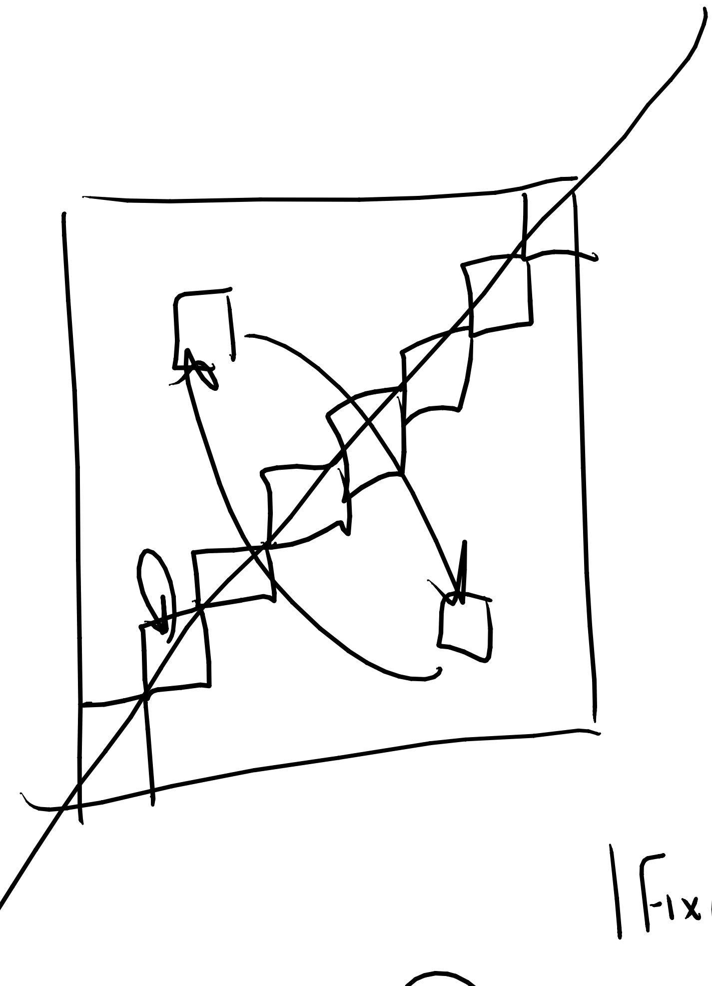

$$|\mathrm{Fix}(b)| = 2^{n^2/2}$$

$p:$ 

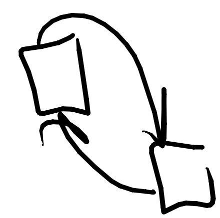


$$|\mathrm{Fix}(p)| = 2^{n^2/2}$$

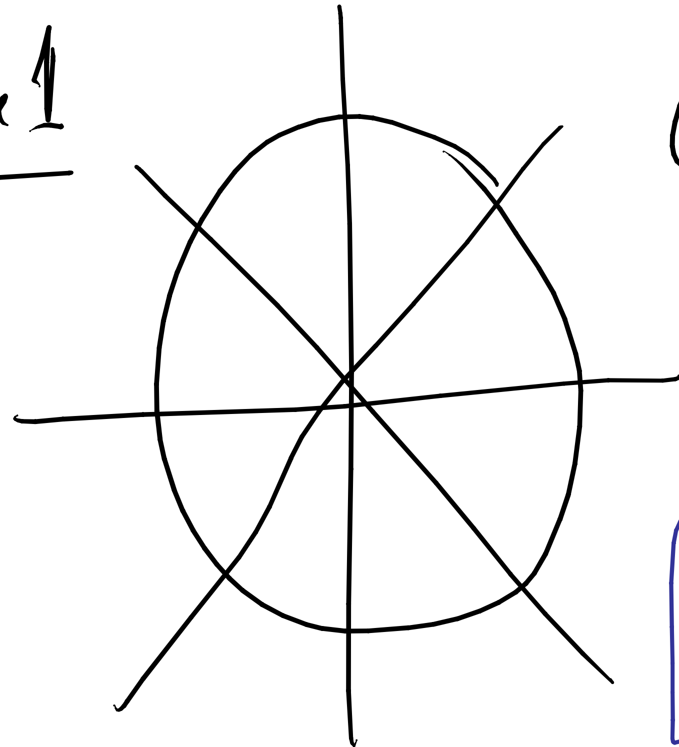
$r =$ 



$|\mathrm{Fix}(r)| =$

$n$  $\frac{n^2-n}{2}$  $2^{n+\frac{n^2-n}{2}}$

## Example 1

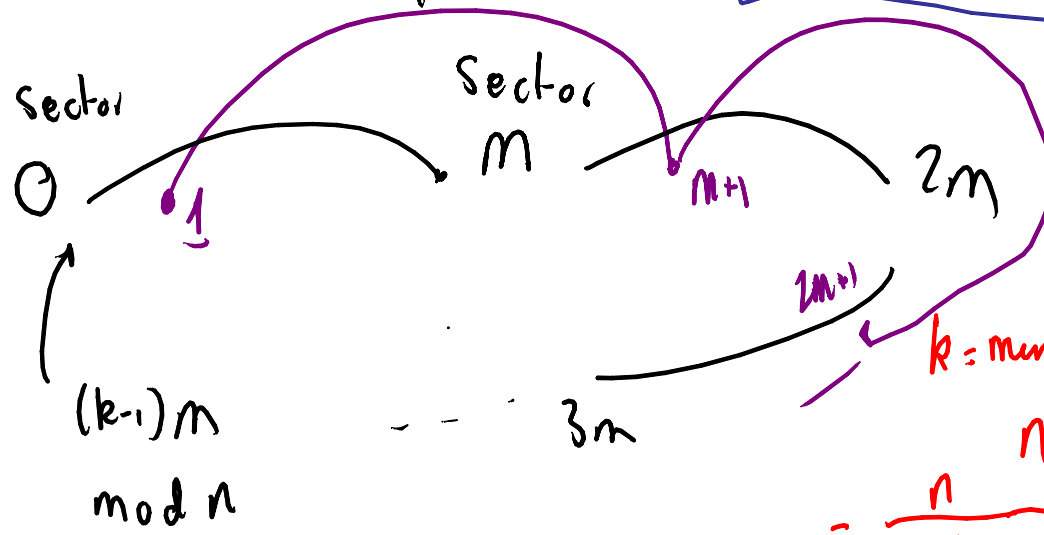


$G = \{e_0, e_1, \dots, e_{n-1}\}$

$e_m(i) = i + m \mod n$

$\boxed{\text{So there are } a_m = \gcd(m,n) \text{ cycles}}$

$e_m$




$k = \min \ell$ s.t. $n$ divides $\ell m$

$= \dfrac{n}{\gcd(m,n)}$

$$\# \text{orbits} = \frac{1}{n} \sum_{m=0}^{n-1} 2^{a_m}$$

$q^{a_m}$

if $q$ colors.

# Cycles of a permutation

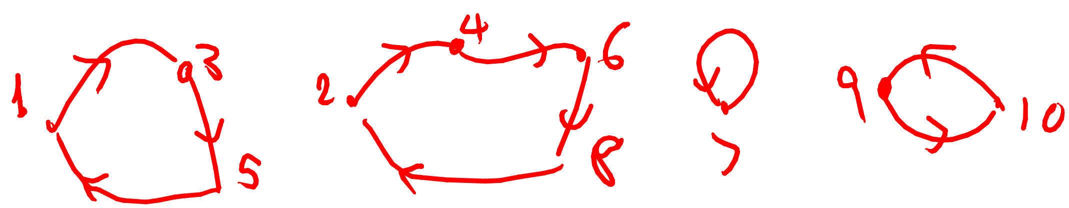
$g : [n] \longrightarrow [n]$ is a permutation

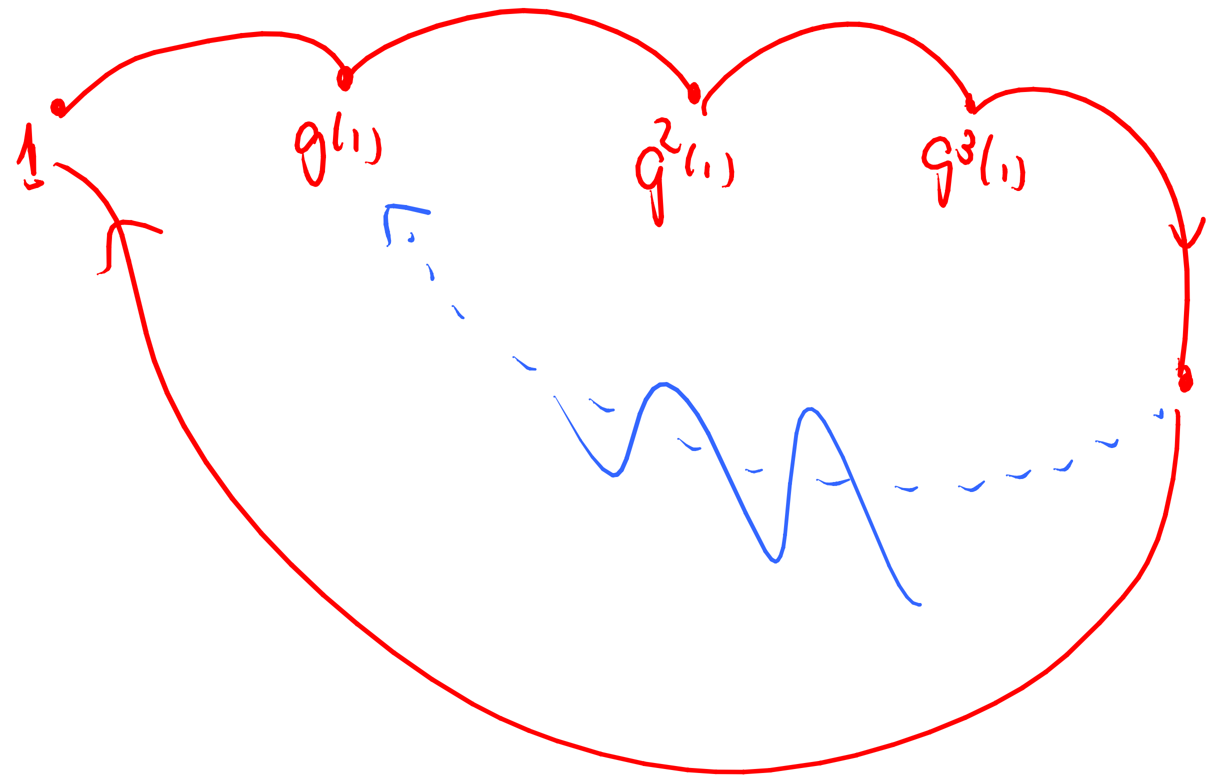
$D_g = \left( [n], \{ (j, g(j)), 1 \leq j \leq n \} \right)$ = union of disjoint cycles

$n = 10$

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $g(j)$ | 3 | 4 | 5 | 6 | 1 | 8 | 7 | 4 | 10 | 9 |

$D_g$ :




1
g(1)
g^2(1)
g^3(1)

Burnside: count # of colorings

Polya: count # of colorings with

$m_1$ of color 1, $m_2$ of color 2,

...

Slightly change scenario

$D$ = domain = Disk
Chessboard

$X = \{$colorings of $D\}$

$G = \{$permutations acting on $D\}$

How does $G$ act on $X$?

$$g * X = ?$$

i.e. what is $g*x(d)$ $\qquad d \in D$

Two possibilities spring to mind

(i) $\cancel{g*x(d) = x(g(d))}$

or

(ii) $g*x(d) = x(g^{-1}(d))$