11\26\12

Example 1

A disk with n sectors. Each sector is colored Red or Blue. Disk can be rotated. How many distinct colorings.

Example 2

Chesss board. Each square can be colored Red or Blue. Board can be rotated or reflected. How many distinct colorings?

$X = \{\text{set of colorings}\}$

where no rotations etc. are allowed

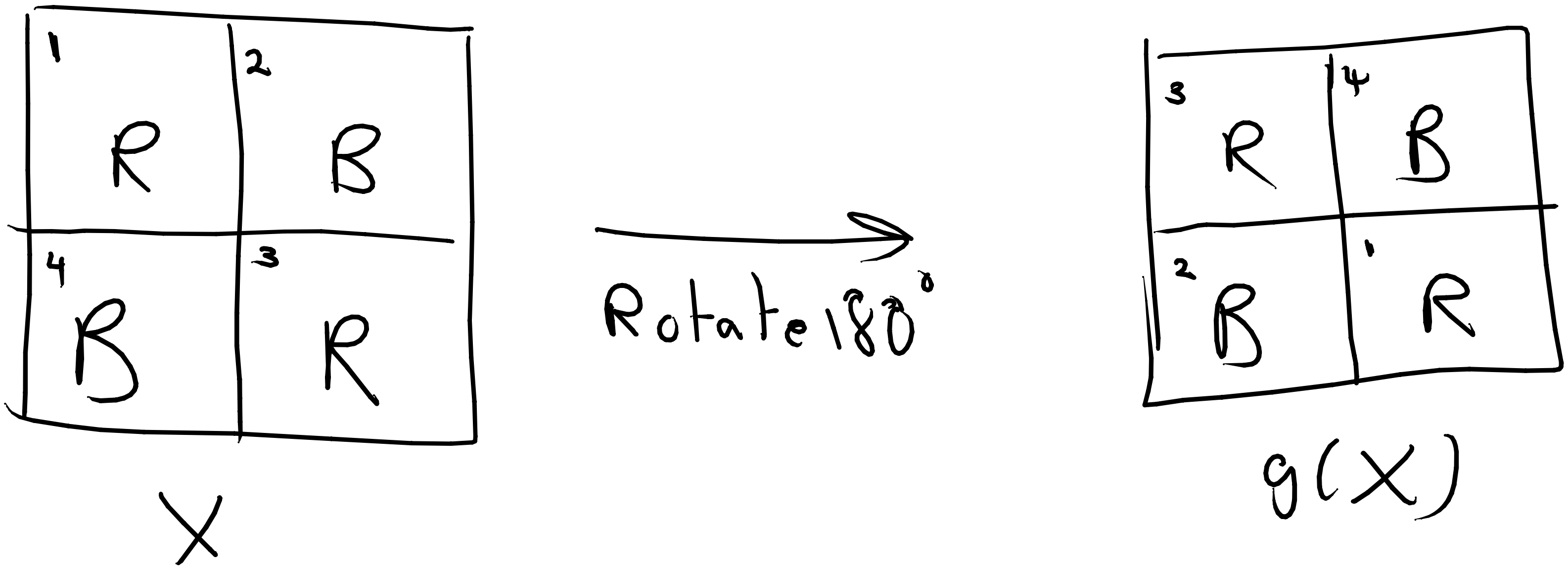
$|X| = 2^n$ in Example 1

$|X| = 2^{n^2}$ in Example 2.

G is a set of permutations acting on X.

$g \in G$ & $x \in X$ then $g(x)$ & x are "same" coloring

g could mean "rotate 90°".



G has a group structure:

(i) $g_1, g_2 \in G \Longrightarrow g_1 \circ g_2 \in G$ composition

(ii) Identity $1_X \in G$

(iii) $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$

(iv) $g \in G \Rightarrow g^{-1} \in G$

Example 1

$$X = 2^{D} \quad \text{where} \quad D = \{0, 1, 2, \dots, n-1\}$$

$$G_1 = \{e_0, e_1, \dots, e_{n-1}\} \quad \text{where} \quad e_i * x = x + i \bmod n$$

$e_i(x)$

Example 2

$$X = 2^{[n]^2}$$

$$G = e, a, b, c, p, q, r, s$$

identity 90° 180° 270°

rotate

reflect

Orbit

$x \in X$, $\mathcal{O}_x = \{ y \in X : \exists g \in G \text{ s.t. } g * x = y \}$

Problem: How many orbits?

Lemma

Orbits partition X.

Proof

$x = 1_X * x$

$x \in \mathcal{O}_x$

$\mathcal{O}_x \cap \mathcal{O}_y \neq \emptyset \Rightarrow \mathcal{O}_x = \mathcal{O}_y$

$\Downarrow$

$\exists g_1, g_2$ s.t.

$g_1 * x = g_2 * y \Rightarrow x = g_1^{-1} g_2 y \Rightarrow g * x = g g_1^{-1} g_2 y$

$\Rightarrow \mathcal{O}_x \subseteq \mathcal{O}_y$

Lemma 2 **Orbit-Stabilizer Theorem**

$S_x = \{ g : g * x = x \}$

stabilizer

If $x \in X$ then $|O_x| |S_x| = |G|$

Proof

Define equivalence relation

$$g_1 \sim g_2 \quad \text{if} \quad g_1 * x = g_2 * x$$

$$\equiv \quad g_1^{-1} g_2 \in S_x$$

Let $A_1, A_2, \ldots, A_m$ be the equivalence classes.

$m = |O_x|$

= # of colorings obtainable from x

We just have to show that

$$|A_i| = |S_x|, \ \forall i$$

Fix i and $g \in A_i$

$$h \in A_i \longleftrightarrow h \in g \circ S_n$$

$$|A_i| = |g \circ S_n| = |S_n|$$

$$g \circ h_1 = g \circ h_2$$

$$\Downarrow$$

$$g^{-1} g h_1 = g^{-1} g h_2$$

$$\Downarrow$$

$$h_1 = h_2$$

$$\#\text{orbits} = \frac{1}{|G|} \sum_{x \in X} |S_x|$$

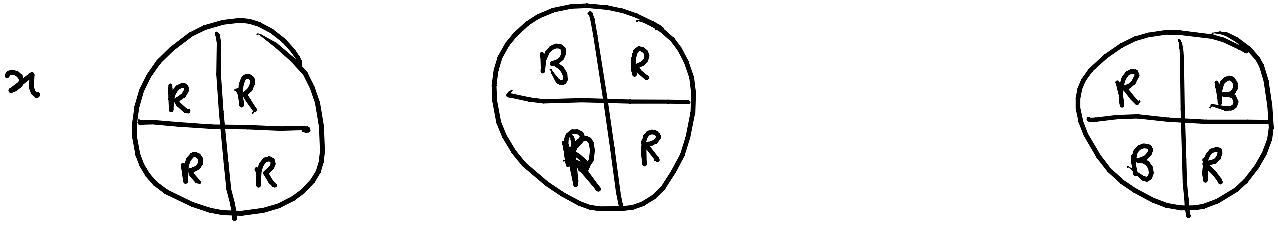
<u>Proof</u>

$$\#\text{orbits} = \sum_{x \in X} \frac{1}{|O_x|}$$

$$= \sum_{x \in X} \frac{|S_x|}{|G|}$$

Example 1

$$\frac{1}{4}\left(4 + 1 + \dots + 2 + \dots\right) = 6$$



Example 2

$$\frac{1}{8}\left(8 + 2 + \dots + 4 + \dots\right) = 6$$