11\19\12

<u>Tic-Tac-Toe</u> – multi-dimensional

The <u>board</u> is $\underbrace{[n] \times [n] \times \cdots \times [n]}_{d \text{ times}}$

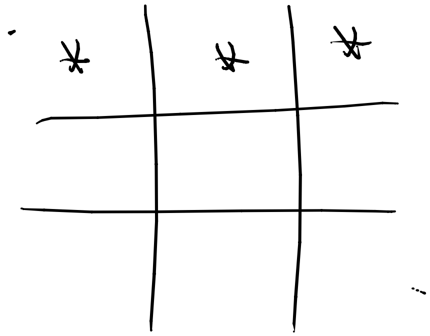
n-vector

A <u>line</u> is a set of points $\{x^{(1)}, x^{(2)}, \ldots, x^{(d)}\}$

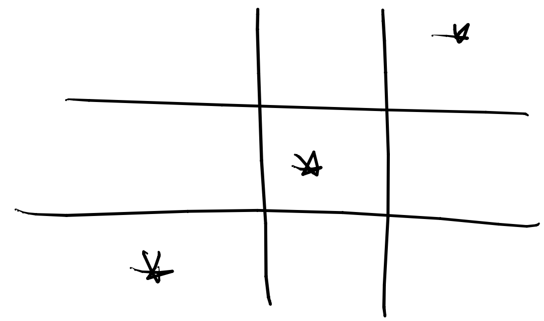
where

not all of this kind

$$x^{(i)} = \begin{cases} (k, k, \ldots, k) & k \in [n] \quad \text{or} \\ (1, 2, \ldots, n) & \text{or} \\ (n, n-1, \ldots, 1) & \end{cases}$$



(1,1) (1,2) (1,3)



(3,1) (2,2) (1,3)

$x^{(1)} = (1,1,1)$

$x^{(2)} = (1,2,3)$

$x^{(1)} = (3,2,1)$

$x^{(2)} = (1,2,3)$

<u>L1</u> # of winning lines $= \dfrac{(n+2)^d - n^d}{2}$

<u>Proof</u> We have to choose $x^{(1)}, x^{(2)}, \dots x^{(d)}$

n+2 choices

$(n+2)^d$

$-n^d$

all constant

$\dfrac{\quad}{2}$

2 ways of getting a line

Two players. Each takes a turn in marking a spot on board O or X and each aims to get a line.

L2: player 2, can only draw the game

Strategy stealing

If player 2 had a winning strategy, then player 1 could use it. Player 1 plays arbitrarily and then follows player 2 strategy.

Pairing Strategy

$[5]^2$ game

$$\begin{bmatrix} 11 & 1 & 8 & 1 & 12 \\ 6 & 2 & 2 & 9 & 10 \\ 3 & 7 & * & 9 & 3 \\ 6 & 7 & 4 & 4 & 10 \\ 12 & 5 & 8 & 5 & 11 \end{bmatrix}$$

Every line contains 2 of the same number.

If P1 plays i then P2 plays the second i

If P1 plays $*$ the P2 plays arbitrarily.

Generalisation

$$\mathcal{F} = \underbrace{A_1, A_2, \ldots, A_N}_{\text{lines}} \subseteq A$$

A move: a player chooses uncolored member of A and gives it their own color.

Player wins if they are the first player to make A_i of that players color

Pairing strategy: $X = \{\underbrace{x_1, x_2, \ldots\ldots, x_{2N-1}, x_{2N}}_{\text{distinct}}\}$

such that

$$A_i \supseteq \{x_{2i-1}, x_{2i}\} \qquad 1 \leq i \leq N$$

2 representative for each set

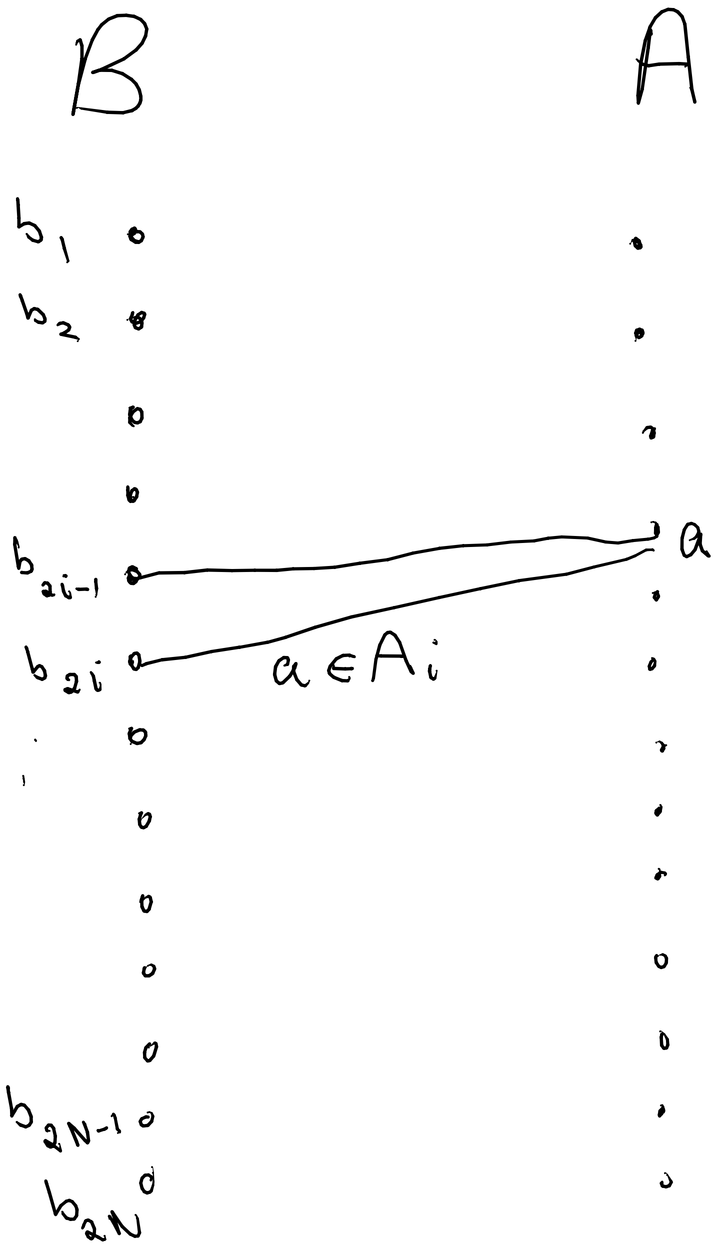
Strategy for P2: if P1 plays $x_{2i-\delta}$

then P2 plays $x_{2i-(1-\delta)}$

L3

$$\left|\bigcup_{i \in S} A_i\right| \geq 2|S| \quad \forall\, S \subseteq [N]$$

then there is a pairing strategy



We want a matching of B into A.

Then $\forall i \ \exists \ a_{2i-1}, a_{2i}$ contained in A_i

Hall's condition reduces to

$$S \subseteq \{1, 2, \dots, N\} \implies$$

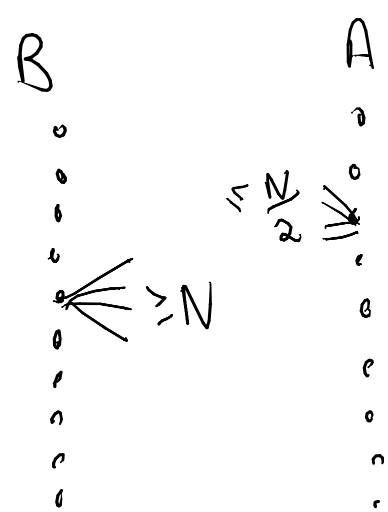
$$\left| \bigcup_{i \in S} A_i \right| \geq 2|S|$$

each $i \in S$ needs to account for b_{2i-1}, b_{2i}

L4

If $|A_i| \geq N$ for $i = 1, 2, \ldots, N$ and each $x \in A$ is in at most $N/2$ sets in $\mathcal{F}$, then there is a pairing strategy.

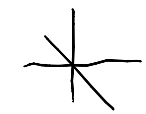
Proof

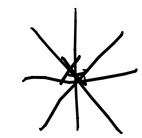


m = #edges

$$|S| \times N \leq m \leq |N(S)| \times \frac{N}{2}$$

Tic Tac Toe $d=2$

n even : each array element in ≤ 3 lines 

n odd : each array element in ≤ 4 lines 

So we immediately see that if

n even & $n \geq 6$ or

n odd & $n \geq 9$

then there is a pairing strategy.

[Cases $n = 4, 7$ settled as draws by other means]

In general

n odd and $n \geq 3^d - 1$ or
n even and $n \geq 2^d - 1$ $\Rightarrow$ pairing strategy

Proof

n odd : # lines through any point $(c_1, c_2, \dots, c_d)$ is $\leq \frac{3^d - 1}{2}$

Divide by two because each line has two orientations

≤ 3 choices for how line continues in each coordinate – up, down, constant

-1 for all constant

n even: if point is $(c_1, c_2, \dots, c_d)$

and we fix where line is constant $i \in I$

then value of $c_i, i \notin I$ determines direction

of line.

E.g. if $c_1 = x \leq n/2$ and $1 \notin I$

then line must go up.

choices for I is $2^d - 1$.

Must also divide by 2.

Quasi-probabilistic Method

If $|A_i| \geq n$ and $N < 2^{n-1}$ then $P2$ can force a draw.

Proof

At any time in game C_i is the set of elements that have been colored by player i.

$$U = A \setminus C_1 \cup C_2$$

$$\Phi = \sum_{i : A_i \cap C_2 = \emptyset} 2^{-|A_i \cap U|}$$

= Expected number of sets of color $\underline{1}$ if both play randomly

P2 keeps $\Phi < 1$ so that at the end, there are no sets of color 1

After P1 first move

$$\Phi < 2^{n-1} \cdot \left(\frac{1}{2}\right)^{n-1} < 1$$

If P2 can keep $\Phi < 1$ then the game is drawn since when $U = \emptyset$, $\Phi = \# i : A_i \subseteq C_1$

Keep track of Φ after choices

$x_1 y_1 x_2 y_2 \cdots x_{k-1} y_{k-1} x_k$

x_i = i'th choice of P1

y_i = i'th choice of P2

Φ_k = value of Φ here

$$\Phi_{k+1} - \Phi_k = -\sum_{\substack{i : A_i \cap C_2 = \emptyset \\ y_k \in A_i}} 2^{-|A_i \cap U|} + \sum_{\substack{i : A_i \cap C_2 = \emptyset \\ y_k \notin A_i \,\&\, x_k \in A_i}} 2^{-|A_i \cap U|}$$

$$< -\sum_{\substack{i : A_i \cap C_2 = \emptyset \\ y_k \in A_i}} 2^{-|A_i \cap U|} +$$

$$+ \sum_{\substack{i : A_i \cap C_2 = \emptyset \\ x_k \in A_i}} 2^{-|A_i \cap U|}$$

So if P2 chooses y_k to

maximise $$\sum_{\substack{i: A_i \cap C_2 = \emptyset \\ y_k \in A_i}} 2^{-|A_i \cap U|}$$

then $\Phi_{k+1} \leq \Phi_k < 1$