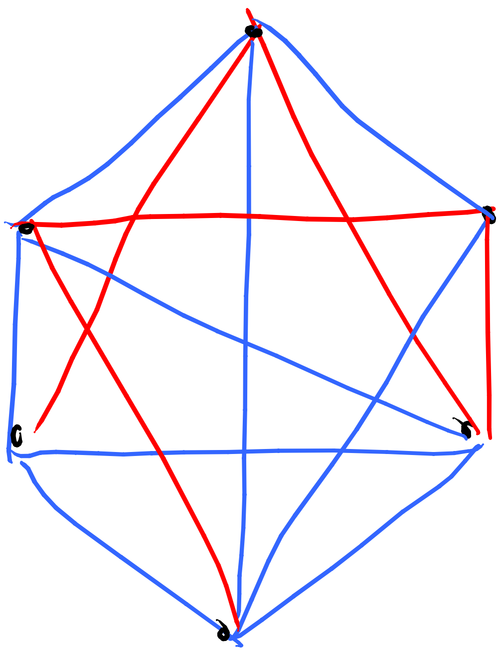
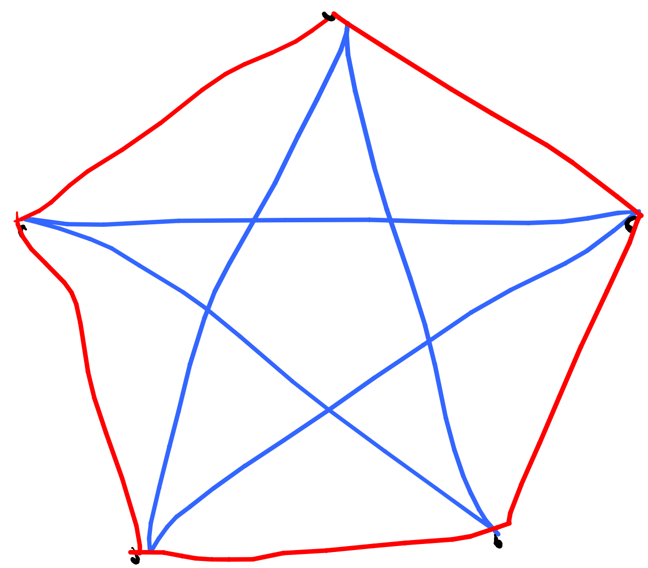
10/29/12

Ramsey Theory

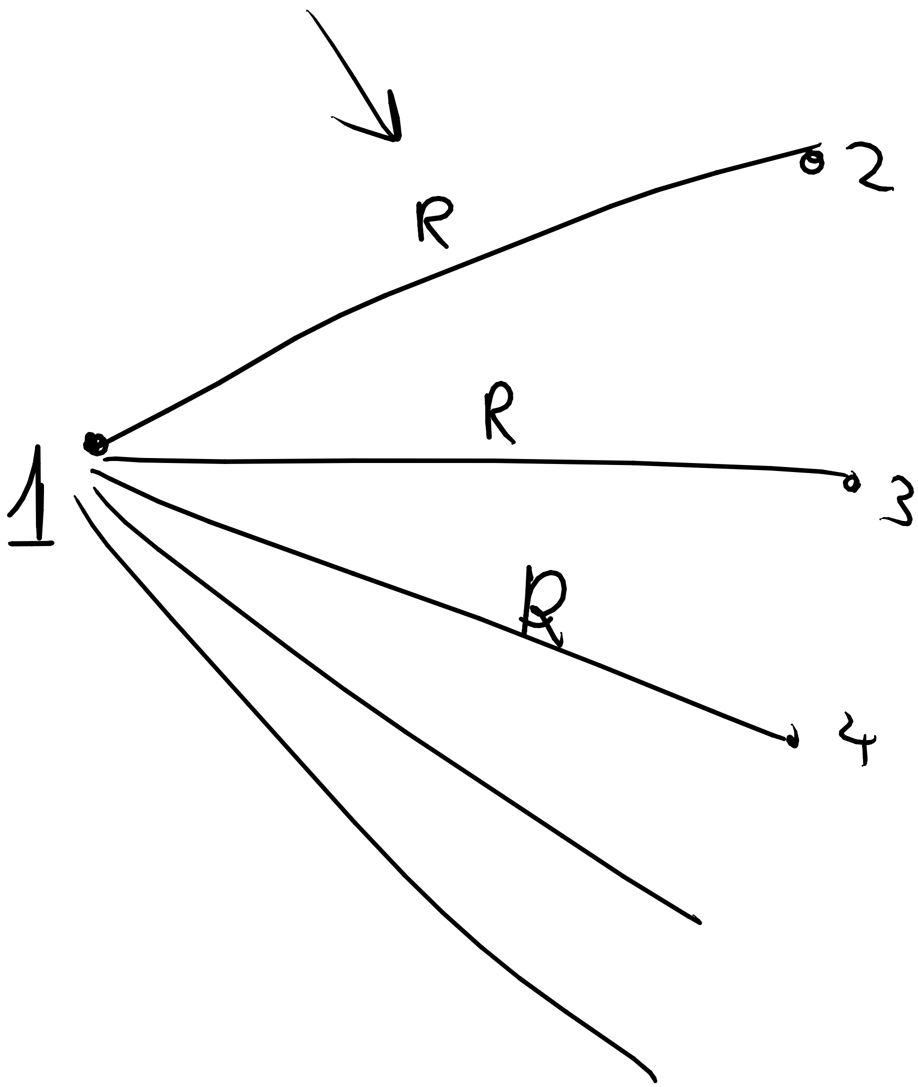


In every 2-coloring of the edges of K_6, there is a mono-chromatic triangle



Coloring of K_5 without a monochromatic triangle

One color used ≥ 3 times



(I) One of 23, 24, 34 is Red

$\Downarrow$

Red Δ

(II) 23, 24, 34 all Blue

$\Downarrow$

Blue Δ

Ramsey's Theorem

For all positive integers k, l, there exists an integer $R(k,l)$ such that if $n \geq R(k,l)$ and the edges of K_n are colored Red or Blue then $\exists$ (i) a Red K_k or a Blue (ii) K_l or both.

[K_l is Blue if all its edges are Blue]

$$R(1,k) = R(k,1) = \underline{1}$$

$$R(2,k) = R(k,2) = k$$

$$R(3,3) = 6$$

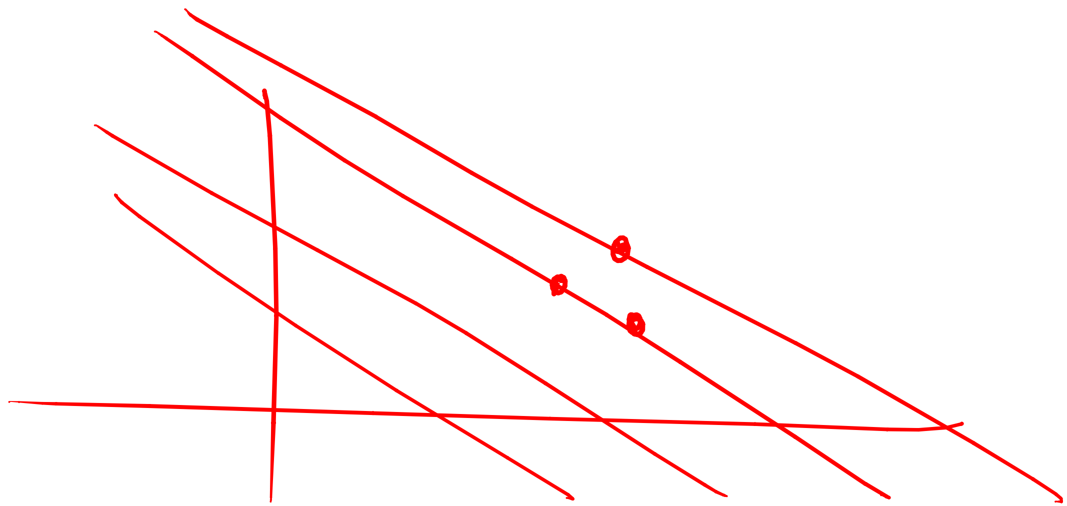
$$R(4,4) = 18$$

$$R(5,5) = ?$$

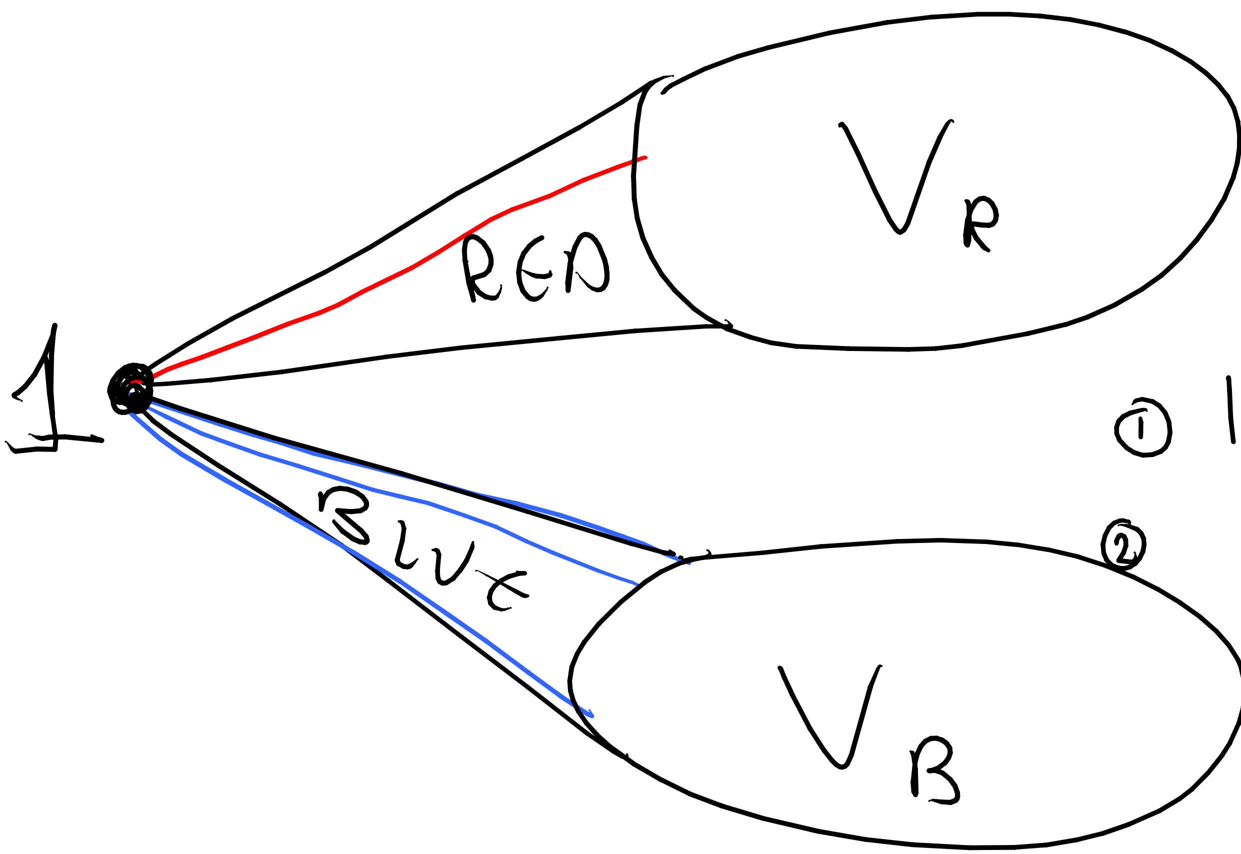
$$R(k,l) \le R(k,l-1) + R(k-1,l)$$

exists $\longleftarrow$ exists & exists

Use induction on $k+l$



$$N = R(k, l-1) + R(k-1, l)$$



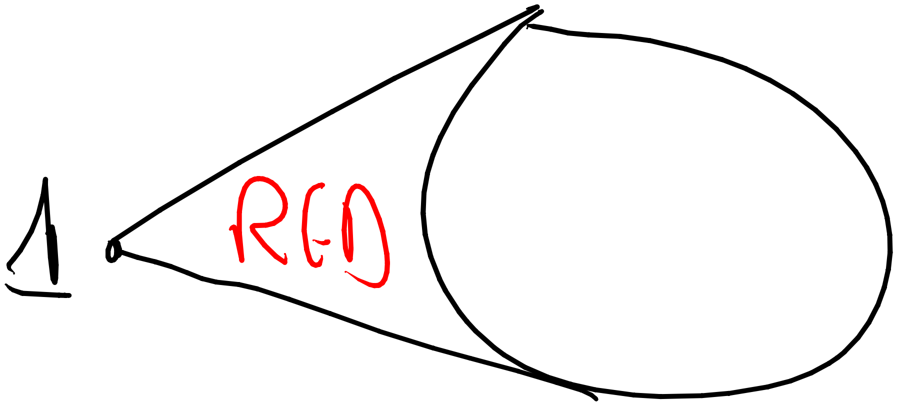
$|V_R| + |V_B| = N - 1$

So, either

① $|V_R| \geq R(k-1, l)$ o

② $|V_B| \geq R(k, l-1)$

Assume for example that $|V_k| \geq R(k-1, \ell)$

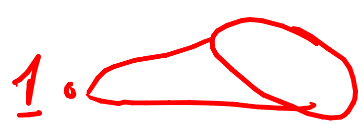


$\Rightarrow$

(i) $\exists$ Blue K_ℓ inside V_k

or

(ii) $\exists$ Red K_{k-1} inside V_k

Red K_k $\longrightarrow$  $\longleftarrow$ K_{k-1}

$$R(k,l) \le \binom{k+l-2}{k-1}$$

Proof

Induction on $k+l$. True for $k+l \le 5$

$$R(k,l) \le R(k,l-1) + R(k-1,l)$$

$$\le \binom{k+l-3}{k-1} + \binom{k+l-3}{k-2} \quad \text{induction}$$

$$= \binom{k+l-2}{k-1} \quad \text{Pascal } \triangle$$

$$R(k,k) < \binom{2k-2}{k-1} < 4^k$$

$$R(k,k) > 2^{k/2}$$

Proof

We show that if $n \leq 2^{k/2}$ then there exists a coloring of K_n without a monochromatic K_k.

We can assume $k \geq 4$ since $R(3,3) = 6 > 2^{3/2}$.

We show that a random 2-coloring has no mono clique, with positive probability.

$\mathcal{E}_R = \{ \exists \text{ Red } k\text{-clique} \}$

$\mathcal{E}_B = \{ \exists \text{ Blue } k\text{-clique} \}$

$C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be an enumeration of the k-cliques of K_n

$\mathcal{E}_{R,i} = \{ C_i \text{ is Red} \}$, $\mathcal{E}_{B,i} = \{ C_i \text{ is Blue} \}$

To show $\Pr(\mathcal{E}_R \cup \mathcal{E}_B) < 1$.

$$\Pr(\mathcal{E}_R \cup \mathcal{E}_B) \le \Pr(\mathcal{E}_R) + \Pr(\mathcal{E}_B)$$

$$= 2\Pr(\mathcal{E}_R) \qquad \text{symmetry}$$

$$= 2\Pr(\mathcal{E}_{R,1} \cup \mathcal{E}_{R,2} \cup \dots \cup \mathcal{E}_{R,N})$$

$$\le 2 \sum_{j=1}^{N} \Pr(\mathcal{E}_{R,j})$$

$$= 2 \sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$= 2 \times N \times \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$< 2 \times \frac{n^k}{k!} \times \frac{1}{2^{k(k-1)/2}}$$

$$\le 2 \times \frac{2^{k^2/2}}{k!} \times \frac{1}{2^{k(k-1)/2}}$$

$$= \frac{2^{1+k/2}}{k!} < 1.$$

Generalisation:

① More colors: $R(k, l, m)$

3 color $N \geq R(k, l, m)$ then $\exists$

Red K_k, or Blue K_l or Green K_m

Existence:

$$R(k, \underbrace{R(l, m)}) \text{ exists}$$

Think of Blue or Green as one color. — Purple.

(11) So far we consider coloring the edges of $K_n \equiv$ coloring 2-element subsets of $[n]$

We could color the r-element subsets of $[n]$

$R(a_1, a_2, \ldots, a_s : r)$

s-color the r-element subsets of $[n]$ $\Rightarrow$ $\exists\ 1 \le i \le s$ & $S \subseteq [n]$ such all r-element subsets of S get color i