10\26\12

$\boxed{x_1}$ $\boxed{x_2}$ $\boxed{x_3}$ - - - $\boxed{x_m}$

$$x_1 + x_2 + \cdots x_m = n$$

$x_1 \cdots x_n$ are integers

Then there exists j such

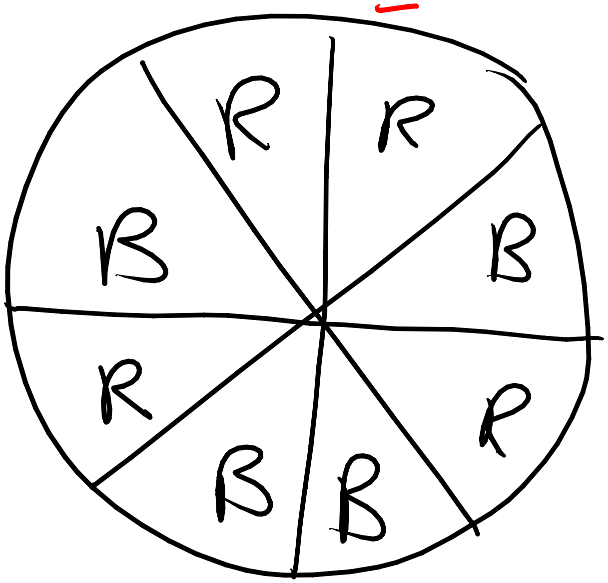
$$x_j \geq \left\lceil \frac{n}{m} \right\rceil$$

Max $\geq$ average

$n = m+1 \Rightarrow \exists\, x_j \geq 2$.

2 disks

1

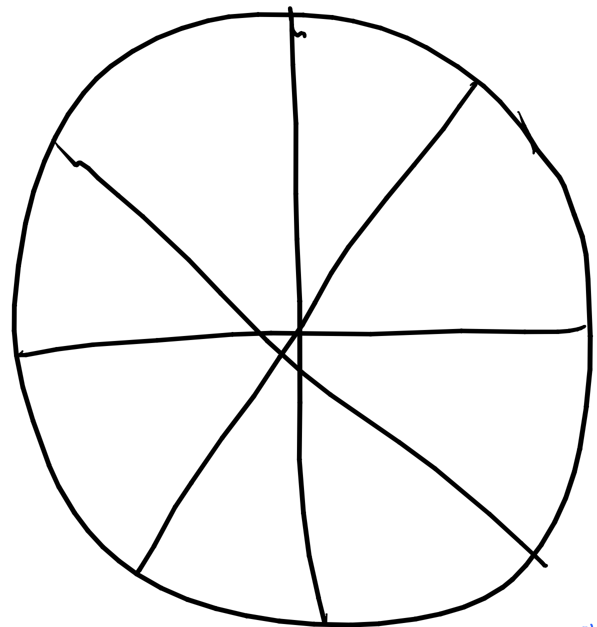


200 sectors

100 Red sectors

100 Blue sectors

2



200 sectors

Sectors are colored arbitrarily

Can place disk 1 on top of disk 2.

Sector bdries match.

$\exists$ 100 sectors one above other, with same color

There are 200 ways of placing disk 1 on top of disk 2.

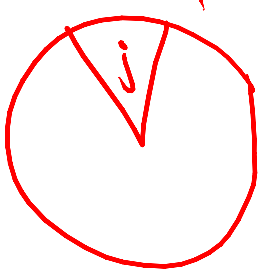
q_i = # of matches in ith position

<u>Claim</u>

$$q_1 + q_2 + \cdots + q_{200} = 200 \times 100$$

$\Rightarrow \exists\, i$ s.t. $q_i \geq 100$.

fixed



i

1 if there is a match.

q_i 1's in row i

each column has 100 1's.

$$\begin{array}{cc} & j \\ i & \left[\ \text{1 if sector } j \text{ is a match for position } i.\ \right] \end{array}$$

Alternative solution

Place disk 1, randomly on top of disk 2.

$$X_j = \begin{cases} 1 & - \text{ color match in sector } i \\ 0 & - \text{ otherwise} \end{cases}$$

$X =$ # matches

$$E(X) = E(X_1) + \dots + E(X_{200})$$
$$= \frac{1}{2} + \dots + \frac{1}{2}$$
$$= 100$$

$\Rightarrow \exists$ a position with ≥ 100 matches

Erdős-Szekerés Theorem

$a_1, a_2, \ldots \quad a_{k^2+1}$ are arbitrary real numbers.

A sequence $a_{i_1}, a_{i_2}, \ldots \quad a_{i_p}$

where $i_1 < i_2 < \ldots < i_p$

is monotone increasing if $a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_p}$

" decreasing if $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_p}$

Theorem

$\exists$ a monotone sequence of length $\geq k+1$.

Let $\ell^{\nearrow}$ = length of longest monotone increasing sequence

$\ell^{\searrow}$ = length of longest monotone decreasing

$$\Longrightarrow \quad \ell^{\nearrow} \times \ell^{\searrow} \geq k^2 + 1$$

3, 8, 2, 4, 5, 1, 9, 4, 2, 3

$\ell^{\nearrow} = 4$, $\ell^{\searrow} = 4$

L_i = length of longest monotone increasing sequence that begins with a_i

$3, \underbrace{2, 4, 5, 2, 7}$ $\qquad L_2 = 4$

$L_i = j$

Put i into box j if $L_i = j$

$\exists$ a box with at $\left\lceil \frac{k^2+1}{\ell\uparrow} \right\rceil$ indices.

The indices in this box define a decreasing sequence

Example

1	2	3	4	5	6	7	8	9	10
6,	3,	8,	2,	4,	5,	6,	7,	2,	5
3	5	1	5	4	3	2	1	2	1

L

$$L_{i_1} = L_{i_2} = \dots = j$$

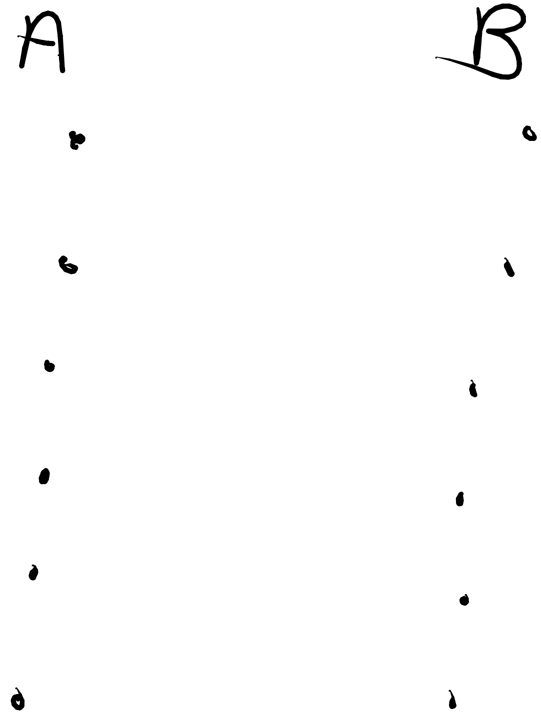
$$i_1 < i_2 < i_3 \dots$$

$$j$$

Suppose $a_{i_t} < a_{i_{t+1}}$

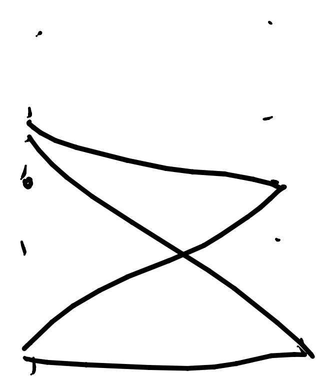
$$\Downarrow$$

$L_{i_t} \geq j+1$ contradiction.



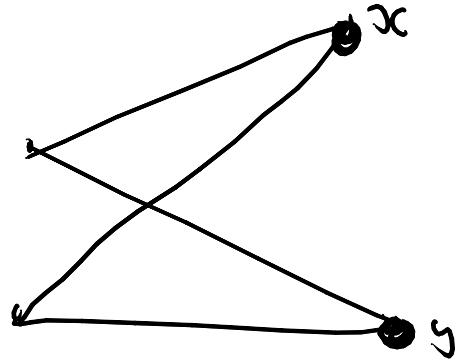
Bipartite graph G
$n + n$ vertices
and m edges

If $m \geq n^{3/2} + n$
then G contains



a copy of C_4.

Looking for



Looking for x, y that have 2 common nbrs.

For each $b \in B$ there $\binom{d(b)}{2}$ pairs of adjacent vertices in A.

If $\nexists$ then

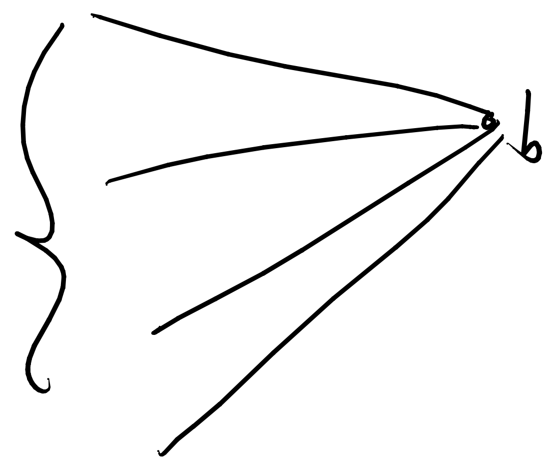
$$\sum_{b \in B} \binom{d(b)}{2} \leq \binom{n}{2}$$

← # pairs of vertices in A

PHP (see next page)

otherwise $\exists\, b, b'$ with a common pair

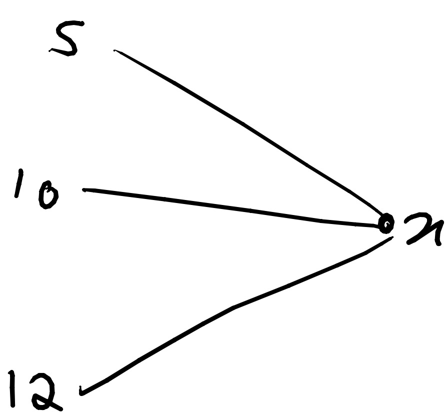
pairs $= \binom{d(b)}{2}$

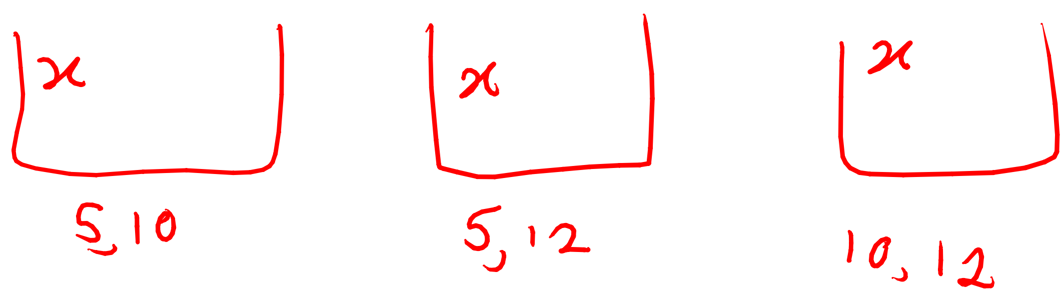


$d(b)$ = degree

Pigeonholes

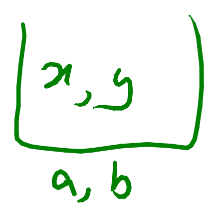
One pigeonhole for every pair of vertices in A

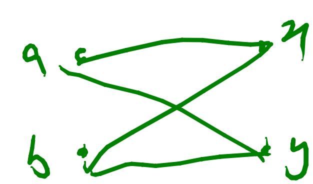


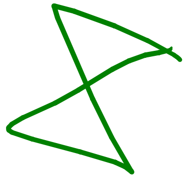


Pigeon-hole with two elements

$\Rightarrow C_4$





With no  we have $\displaystyle\sum_{b \in B} \binom{d(b)}{2} \leq \binom{n}{2}$

We know that $\displaystyle\sum_{b \in B} d(b) = m \Rightarrow \quad \geq n\binom{m/n}{2}$

So $n\binom{m/n}{2} \leq \binom{n}{2}$

$\Rightarrow \cancel{n}(m/n)(m/n - 1) \leq \cancel{n}(n-1)$

This fails if $m \geq n^{3/2} + n \qquad (n^{1/2})^2 > n-1$

Finally

$$\frac{1}{2}\sum_{b\in B}\left[d(b)^2 - d(b)\right]$$ is minimised at $d(b) = \frac{m}{n}$

$$= \frac{1}{2}\sum_{b\in B} d(b)^2 - \frac{1}{2}m$$

Minimise $x_1^2 + x_2^2 + \dots + x_n^2$

s.t. $x_1 + \dots + x_n = m$

Take any solution with $x_1 < x_2$ say.

$$x_1^2 + x_2^2 > \left(\frac{x_1+x_2}{2}\right)^2 + \left(\frac{x_1+x_2}{2}\right)^2$$

$$x_1^2 + x_2^2 > \frac{x_1^2}{2} + \frac{x_2^2}{2} + x_1x_2 \equiv \frac{(x_1-x_2)^2}{2} \geq 0$$