G

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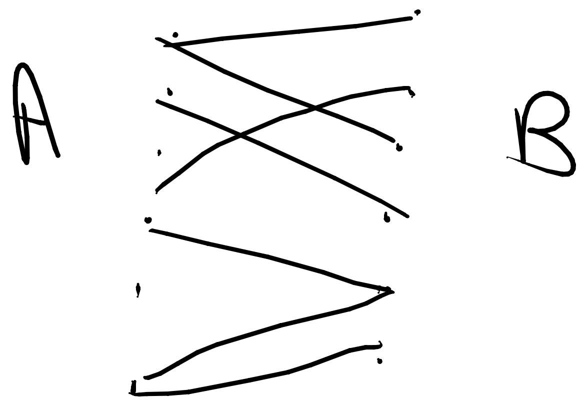


Hall's Theorem

G contains a matching of size $|A|$ iff

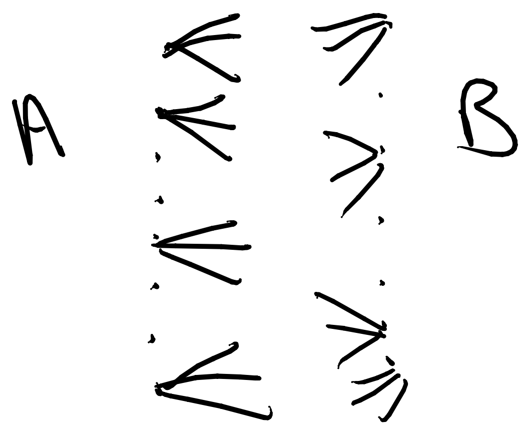
$|N(S)| \geq |S|$, $\forall S \subseteq A$.

Marriage Theorem

Suppose that G is a k-regular bipartite graph $(k \geq 1)$.

[All vertices have degree k]

Then G contains a perfect matching.



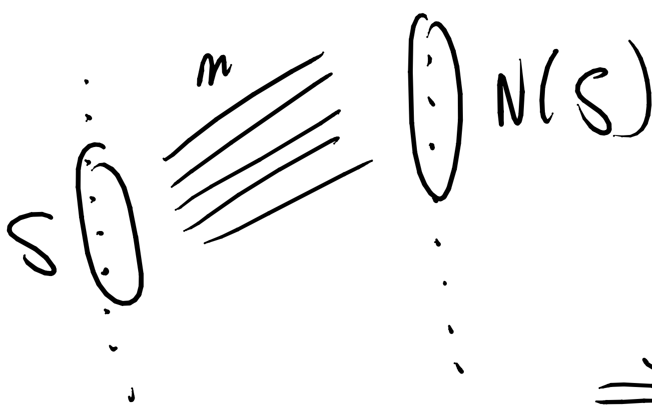
edges in G is

$$k|A| = k|B|$$

$$\Rightarrow |A| = |B|$$

So a matching of size $|A|$ is perfect.

Check Hall's condition



$m = \#$ edges $S : N(S)$

$$k|S| = m \le k|N(S)|$$

$$\Rightarrow |S| \le |N(S)|.$$

In fact if G is k-regular then

edges $E = M_1 \cup M_2 \cup \cdots \cup M_k$

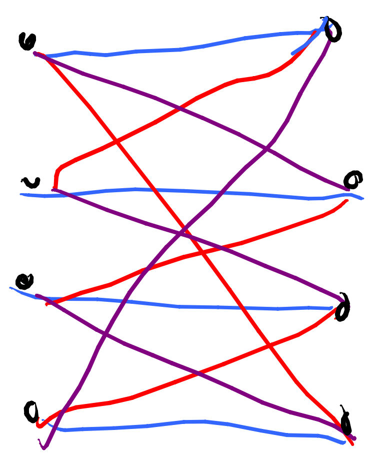
where each M_i is a perfect matching

and $M_i \cap M_j = \emptyset$.

G contains one perfect matching M, say.

$G - M$ is $(k-1)$-regular — apply induction.

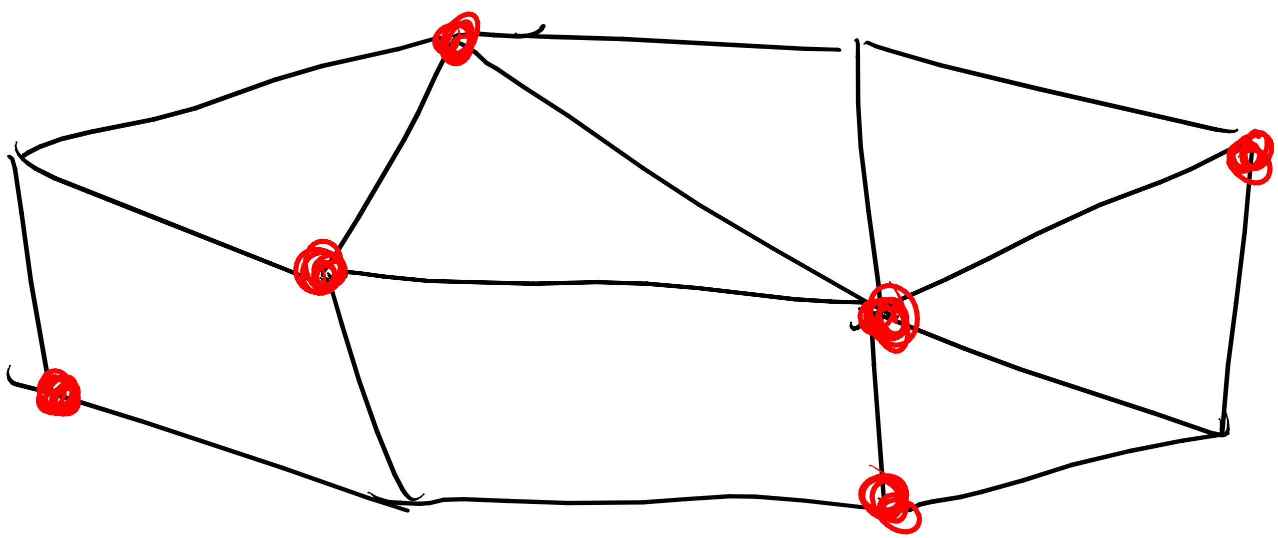
This implies that the edges of k-regular bipartite G can be colored with k colors so that each vertex is incident with edges of each color



Give each M_i a different color

Edge Covers - Konig's Theorem

A set of vertices $X \subseteq V$ is a cover if every edge contains a vertex from X.

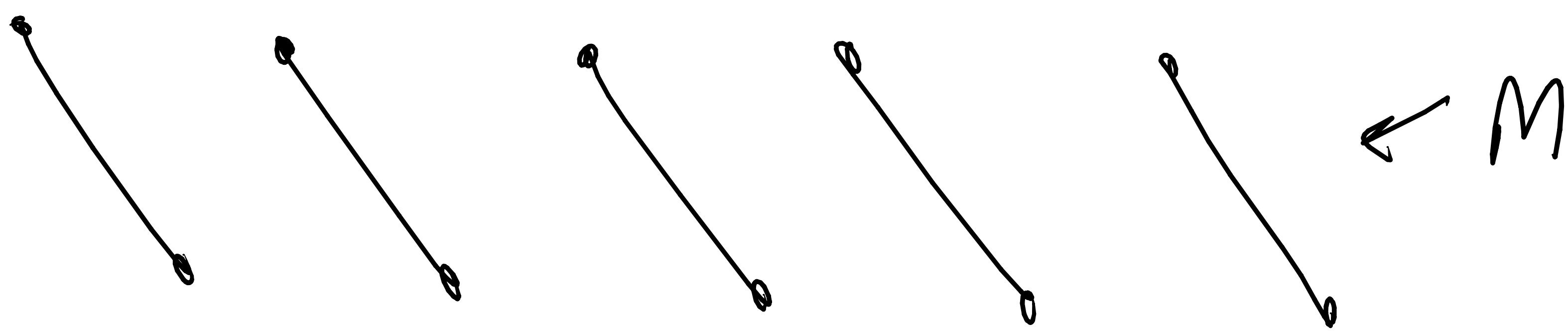


● is a cover.

Lemma

If X is a cover and M is a matching then $|X| \geq |M|$.

Proof



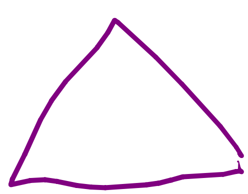
Need a different vertex for each $e \in M$

$$\overbrace{\max_{M} \underset{\substack{\uparrow \\ \text{matching}}}{|M|}}^{\mu} \leq \overbrace{\min_{X} \underset{\substack{\uparrow \\ \text{cover}}}{|X|}}^{\beta}$$

Theorem (König)

If G is bipartite then $\mu = \beta$.

Note, not true for non-bipartite



$\mu = 1$

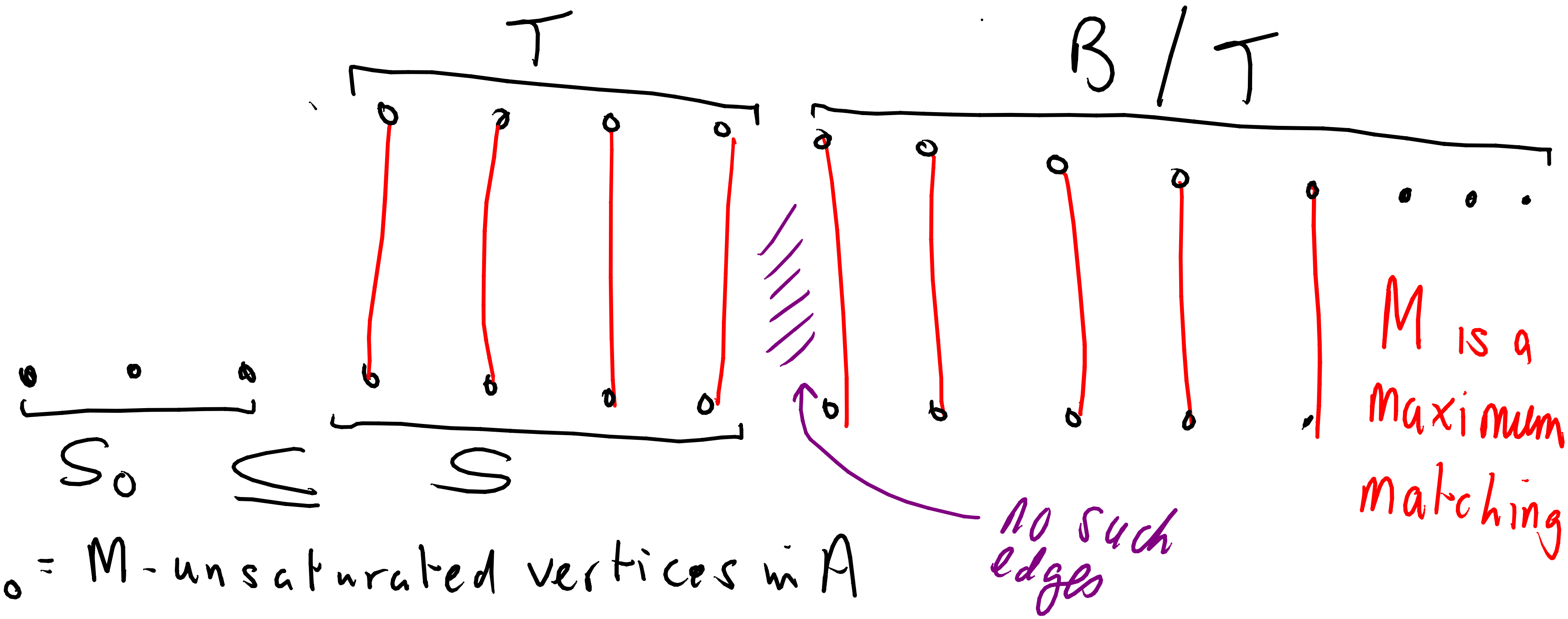
$\beta = 2$

Proof of König's Theorem

Let M be a maximum matching.

We find a cover of size $|M|$.

$X = T \cup (A \setminus S)$ is a cover. & $|X| = |M|$.



S_0 = M-unsaturated vertices in A

S = vertices in A reachable by alternating path from S_0