21-301 Combinatorics
Homework 9
Due: Monday, December 5

1. There are two boxes. Initially, one box contains m chips and the other contains n chips. Such a position is denoted by (m, n) where $m, n > 0$. A move consists of emptying one of the boxes and dividing the contents of the other between the two boxes with at least one chip in each box. There is a unique terminal position, $(1, 1)$. Find all P positions. Hint: Compute the N, P positions for small m, n and see if you can see the pattern.

   Solution: Let $S = \{(m, n) : \text{both } m \text{ and } n \text{ are odd}\}$. We will show that S is the set of losing positions. If $(m, n) \notin S$ then at least one of m, n is even. Suppose that m is even. Then we take n chips from one box and put m_1, m_2 into the two boxes where $m_1 + m_2 = m$ and both m_1, m_2 are odd. If $(m, n) \in S$ and $(m, n) \neq (1, 1)$ then whichever box we empty, we have an odd number of chips to split and we cannot split into two odd numbers i.e. we cannot place the next player in an S-posiiton.

2. Consider the following take-away game: In the first move you are not allowed to take the whole pile. After that, if a player removes x chips, then the next player can remove up to $\lfloor 3x/2 \rfloor$ chips. Determine the P positions.

   Solution: The P-positions, $\{H_1, H_2, \ldots,\}$ satisfy the recurrence

$$H_{j+1} = H_j + H_k \text{ where } k = \min_{0 \le \ell \le j} \{\ell : H_j \le \lfloor 3H_\ell/2 \rfloor\}. \tag{1}$$

   The first 8 values are given by

j	0	1	2	3	4	5	6	7
H_j	1	2	4	8	16	32	64	128

   We can see that $H_j = 2^j$, but we must prove this by induction. But this follows from

$$\lfloor 3 \times 2^{j-1}/2 \rfloor < 2^j$$

   which implies that $k = j$ in (1).

3. Find the Sprague-Grundy numbers $g(n)$ for the take-away games with subtraction sets

   (a) $S = \{1, 3, 5\}$.

   (b) $S = \{1, 3, 6\}$.

   (c) Suppose now that there are two piles and the rules for each pile are as above. Now find the P positions for the two pile game.

Solution:

(a) The first few numbers are

j	0	1	2	3	4	5	6	7	8	9	10
$g_1(j)$	0	1	0	1	0	1	0	1	0	1	0

It is apparent that $g_1(j) = j \mod 2$ and this follows by an easy induction: If j is even then $j - x, x \in S$ is odd and if j is odd then $j - x, x \in S$ is even.

(b) The first few numbers are

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$g_2(j)$	0	1	0	1	0	1	2	3	2	0	1	0	1	0	1	2	3	2

j	18	19	20	21	22	23	24	25	26
$g_2(j)$	0	1	0	1	0	1	2	3	2

So, we see that the pattern 0 1 0 1 0 1 2 3 2 repeats itself. Again, induction can be used to verify that this continues indefinitely.

(c) The P-positions are those j for which $g_1(j) \oplus g_2(j) = 0$. Thus $P = \{j : j \mod 18 \leq 5\}$.