21-301 Combinatorics
Homework 8
Due: Monday, November 14

1. Given any sequence of n integers, positive or negative, not necessarily all different, show that some consecutive subsequence has the property that the sum of the members of the subsequence is a multiple of n.

   Solution: Let the sequence be $x_1, x_2, \ldots, x_n$ and let $s_i = x_1 + \cdots + x_i \mod n$ for $i = 1, 2, \ldots, n$. If there exists i with $s_i = 0$ then n divides $x_1 + \cdots + x_i$. Otherwise, $s_1, s_2, \ldots, s_n$ all take values in $[n-1]$. By the pigeon-hole principle, there exist $i < j$ such that $s_i = s_j$ and then n divides $x_{i+1} + \cdots + x_j$.

2. Suppose that $a_1, a_2, \ldots, a_n \in [n]$ and $b_1, b_2, \ldots, b_n \in [n]$. An interval I is a set of the form $\{i, i+1, \ldots, j\}$. Let $a_I = a_i + a_{i+1} + \cdots + a_j$ and $b_I = b_i + b_{i+1} + \cdots + b_j$. Show that there exist intervals I, J such that $a_I = b_J$.
   Hint: Assume that $\sum_{i=1}^n a_i \geq \sum_{i=1}^n b_i$. Then show that for each k there exists j and $R_k \in [0, n-1]$ such that $\sum_{s=1}^j a_s = \sum_{t=1}^k b_t + R_k$.

   Solution: Now, for any k it is always possible to express $\sum_{i=1}^k b_i$ as $\sum_{i=1}^k b_i = \sum_{i=1}^j a_i + R_k$ $R_k \in [0, n-1]$ for some $j = j(k) \in [0, n]$. Indeed, either $\sum_{i=1}^k b_i < a_1$ and then we can take $j = 1$ and directly see that $R_k = a_1 - \sum_{i=1}^k b_i \in [0, n-1]$. Otherwise, take the largest j such that $S_k = \sum_{i=1}^k b_i - \sum_{i=1}^j a_i \geq 0$. If $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ then there is nothing to prove and so we assume that $\sum_{i=1}^n a_i > \sum_{i=1}^n b_i$. This implies that $j < n$. If $S_k \geq n$ then $S_{k+1} \geq 0$, contradiction.

   Consider the set of values R_k, for $k \in [1, n]$. If $R_k = 0$, for some k then we are done since $\sum_{i=1}^k b_i = \sum_{i=1}^{j(k)} a_i$. If not, there are only $n-1$ possible values for the n quantities $R_1, \ldots, R_n$ and so there exist $k_1 < k_2$ such that $R_{k_1} = R_{k_2}$. But then

$$\sum_{i=1}^{k_1} b_i - \sum_{i=1}^{j(k_1)} a_i = R_{k_1} = R_{k_2} = \sum_{i=1}^{k_2} b_i - \sum_{i=1}^{j(k_2)} a_i$$

and therefore,

$$\sum_{i=j_1}^{k_2} b_i = \sum_{i=j(k_1)}^{j(k_2)} a_i.$$

3. Prove that if $n \geq R(2k, 2k)$ and if we 2-color the edges of $K_{n,n}$ then there is a mono-chromatic copy of $K_{k,k}$.

   Solution: Given a coloring σ of $K_{n,n}$ we construct a coloring τ of the edges of K_n as follows. If $i < j$ then we give the edge (i, j) of K_n the same color that is given to edge (i, j) under σ.

   Since $n \geq R(2k, 2k)$ we see that K_n contains a mono-colored copy of K_{2k}. If the set of vertices of this copy is S, divide S into two parts S_1, S_2 of size k where $\max S_1 < \min S_2$. It follows that the bipartite sub-graph of $K_{n,n}$ defined by S_1, S_2 is mono-colored under σ.