21-301 Combinatorics
Homework 7
Due: Monday, November 7

1. Let $m = \lfloor n/2 \rfloor$. Describe a family $\mathcal{A}$ of size $2^{n-1} + \binom{n-1}{m-1}$ that has the following property: If $A_1, A_2 \in \mathcal{A}$ are disjoint then $A_1 \cup A_2 = [n]$.

   Solution: If $n = 2m+1$ is odd, let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ where $\mathcal{A}_1 = \{A \subseteq [n] : |A| \geq m+1\}$ and $\mathcal{A}_2 = \{A \subseteq [n] : |A| = m, A \ni 1\}$. Here $|\mathcal{A}_1| = 2^{n-1}$, because if we partition the subsets of $[n]$ into 2^{n-1} pairs, a set and its complement, then the larger of the two sets is in $\mathcal{A}_1$. Clearly $|\mathcal{A}_2| = \binom{n-1}{m-1}$. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ are intersecting families and if $A \in \mathcal{A}_1, B \in \mathcal{A}_2$ and $A \cap B = \emptyset$ then we have $|B| = m$ and $|A| \geq m+1$ and so A, B must be complementary.

   If $n = 2m$ is even then let $\mathcal{A} = \{A \subseteq [n] : |A| \geq m\}$. Now

   $$|\mathcal{A}| = |\mathcal{A}_1| + \binom{n}{m} = |\mathcal{A}_1| + \frac{1}{2}\binom{n}{m} + \frac{1}{2}\binom{n}{m} = 2^{n-1} + \frac{1}{2}\binom{n}{m} = 2^{n-1} + \binom{n-1}{m-1}.$$

   Here, $|\mathcal{A}_1| + \frac{1}{2}\binom{n}{m} = 2^{n-1}$ because we can obtain this number of sets by taking one set from each pair of complementary sets. Each $A \in \mathcal{A}_1$ intersects all sets of size m or more and two sets of size m fail to intersect only when they are complementary.

2. Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be distinct finite subsets of $\{1, 2, 3, \ldots, \}$ such that

   - for every i, $A_i \cap B_i = \emptyset$, and
   - for every $i \neq j$, $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$.

   Prove that for every real number $0 \leq p \leq 1$.

   $$\sum_{i=1}^{n} p^{|A_i|}(1-p)^{|B_i|} \leq 1. \qquad (1)$$

   (Hint: Define disjoint events $\mathcal{E}_i$ such that the LHS of (1) is $\sum_i \Pr(\mathcal{E}_i)$.)

   Solution: Choose a random subset X by including each integer with probability p. Then let

   $$\mathcal{E} = \{A_i \subseteq X \text{ and } B_i \cap X = \emptyset\}.$$

   Then

   $$\Pr(\mathcal{E}_i) = p^{|A_i|}(1-p)^{|B_i|}$$

   and $\mathcal{E}_i, \mathcal{E}_j$ are disjoint for $i \neq j$ since for example $A_i \subseteq X$ and $A_i \cap B_j \neq \emptyset$ implies that $B_j \cap X \neq \emptyset$.

3. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i \geq 1$ for $i = 1, 2, \ldots, n$. Let J be any open interval of width 2. Show that of the 2^n sums $\sum_{i=1}^n \epsilon_i x_i$, ($\epsilon_i = \pm 1$), at most $\binom{n}{\lfloor n/2 \rfloor}$ lie in J.

   (Hint: use Sperner's lemma.)

   Solution: For $A \subseteq [n]$ let $x_A = \sum_{i \in A} x_i - \sum_{i \notin A} x_i$. Let $\mathcal{A} = \{A : x_A \in J\}$. It is enough to show that $\mathcal{A}$ is a Sperner family. Indeed, if $A, B \in \mathcal{A}$ and $A \subset B$ then $x_B - x_A = 2\sum_{i \in B \setminus A} x_i \geq 2$. Thus we cannot have both $x_A, x_B \in J$.