21-301 Combinatorics
Homework 7
Due: Monday, November 7

1. Let $m = \lfloor n/2 \rfloor$. Describe a family $\mathcal{A}$ of size $2^{n-1} + \binom{n-1}{m-1}$ that has the following property: If $A_1, A_2 \in \mathcal{A}$ are disjoint then $A_1 \cup A_2 = [n]$.

2. Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be distinct finite subsets of $\{1, 2, 3, \ldots, \}$ such that

   - for every i, $A_i \cap B_i = \emptyset$, and
   - for every $i \neq j$, $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$.

   Prove that for every real number $0 \leq p \leq 1$.

   $$\sum_{i=1}^{n} p^{|A_i|}(1-p)^{|B_i|} \leq 1. \tag{1}$$

   (Hint: Define disjoint events $\mathcal{E}_i$ such that the LHS of (1) is $\sum_i \Pr(\mathcal{E}_i)$.)

3. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_i \geq 1$ for $i = 1, 2, \ldots, n$. Let J be any open interval of width 2. Show that of the 2^n sums $\sum_{i=1}^{n} \epsilon_i x_i$, ($\epsilon_i = \pm 1$), at most $\binom{n}{\lfloor n/2 \rfloor}$ lie in J.
   (Hint: use Sperner's lemma.)