21-301 Combinatorics
Homework 5
Due: Monday, October 4

1. A clown stands at the side of a swimming pool. In his hand is a bag containing n red balls and n blue balls. At each step he puts his hand into the bag and pulls out a random ball and throws it away. If the ball is red, he makes a step towards the pool and if it is blue, he makes a step away from the pool. What is the probability that the clown falls in to the pool?

   Solution: Imagine that the pool is to the left of the clown and that the sequence of moves by the clown can be descibed by a sequence $\mathbf{x}$ of n R's and L's. The clown will stay dry iff every prefix $x_1x_2\cdots x_k$ of $\mathbf{x}$ has at least as many R's as L's. The number of choices for $\mathbf{x}$ is then the number of choices of grid paths from $(0,0)$ to (n,n) that never go below the diagonal i.e. $\frac{1}{n+1}\binom{2n}{n}$ and the probability of staying dry is

$$\frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}} = \frac{1}{n+1}.$$

2. Suppose $n \geq 4$ and let $E_1, E_2, \ldots, E_m$ be an arbitrary collection of n-subsets of E. Suppose that $m \leq 4^{n-1}/3^n$. Show that there is a 4-coloring of E such that in every E_i, all 4 colors are represented.

   Solution: Color E randomly. Let $\mathcal{E}_i$ be the event that E_i does not receive all four colors. Let $\mathcal{E} = \bigcup_{i=1}^m \mathcal{E}_i$. We need to show that $\Pr(\mathcal{E}) < 1$. But

$$\Pr(\mathcal{E}) \leq \sum_{i=1}^{m} \Pr(\mathcal{E}_i) = m \Pr(\mathcal{E}_1). \tag{1}$$

   Let Ω be the set of colorings of E_1. Let Ω_j be the set of colorings E_1 that do not use color j.

   Now if $\Omega_J = \bigcap_{j \in J} \Omega_j$ then

$$\begin{aligned}
\left| \bigcup_{j=1}^{4} \Omega_j \right| &= \sum_{j=1}^{4} |\Omega_j| - \sum_{\substack{J \subseteq [4] \\ |J|=2}} |\Omega_J| + \sum_{\substack{J \subseteq [4] \\ |J|=3}} |\Omega_J| \\
&= 4 \cdot 3^n - 6 \cdot 2^n + 4 \cdot 1 \\
&< 4 \cdot 3^n.
\end{aligned}$$

   So,

$$\Pr(\mathcal{E}_1) = \frac{\left|\bigcup_{j=1}^4 \Omega_j\right|}{|\Omega|} < \frac{4 \cdot 3^n}{4^n}.$$

   Plugging this into (1) we see that $\Pr(\mathcal{E}) < 1$ as desired.

3. Let $p = 3k+2$ be prime. Show that every set of positive integers S not containing a multiple of p contains a subset T of size at least $|S|/3$ such that if $x, y, z \in T$ then $x + y \neq z \mod p$.

(Hint: Let $C = \{k+1, k+2, \ldots, 2k+1\}$ and let x be chosen randomly from $\{1, 2, \ldots, p-1\}$. Now consider the number of $s \in S$ such that $xs \mod p$ lies in C.)

Solution: We observe first that C is sum free. Let $k+x, k+y, k+z \in C$ where $1 \leq x, y, z \leq k+1$. If $k+x+k+y = k+z \mod p$ then $z = k+x+y \mod p$ which implies that x is at least $k+2$, contradiction.

Next let $S = \{s_1, s_2, \ldots, s_N\}$ and let $Z_i = 1$ if $xs_i \mod p \in C$ and let $Z_i = 0$ otherwise. Then xs_i is equally likely to be any member $\{1, 2, \ldots, p-1\}$. So,

$$\Pr(Z_i = 1) = \frac{k+1}{3k+1} > \frac{1}{3}.$$

So, $E(Z_1 + \cdots + Z_N) > N/3$ and hence there exists an x such that if $T = xS \cap C$ then $|T| > N/3$. But then T is sum-free and $x^{-1}T \mod p$ is a subset of S that is sum-free.