21-301 Combinatorics
Homework 5
Due: Monday, October 17

1. A clown stands at the side of a swimming pool. In his hand is a bag containing n red balls and n blue balls. At each step he puts his hand into the bag and pulls out a random ball and throws it away. If the ball is red, he makes a step towards the pool and if it is blue, he makes a step away from the pool. What is the probability that the clown falls in to the pool?

2. Suppose $n \geq 4$ and let $E_1, E_2, \ldots, E_m$ be an arbitrary collection of n-subsets of E. Suppose that $m \leq 4^{n-1}/3^n$. Show that there is a 4-coloring of E such that in every E_i, all 4 colors are represented.

3. Let $p = 3k + 2$ be prime. Show that every set of positive integers S not containing a multiple of p contains a subset T of size at least $|S|/3$ such that if $x, y, z \in T$ then $x + y \neq z \mod p$.

   (Hint: Let $C = \{k+1, k+2, \ldots, 2k+1\}$ and let x be chosen randomly from $\{1, 2, \ldots, p-1\}$. Now consider the number of $s \in S$ such that $xs \mod p$ lies in C.)