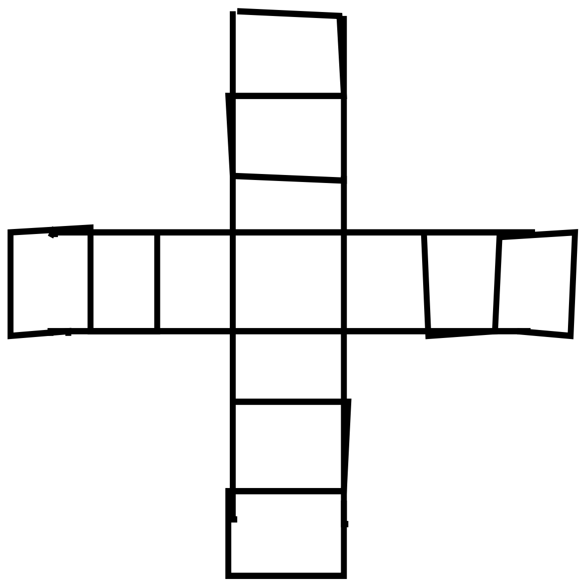
# 21-301 Combinatorics
## Homework 4
## Due: Monday, October 10

1. How many ways are there of coloring the squares of diagram below in $q$ colors if the group acting is $G = \{e, a, b, c, p, q, r, s\}$ of the notes on "Polya Theory of Counting".



You can assume that each arm has $n$ squares.

**Solution:** We get the following cycle structure:

(1) $e$: $(4n+1) \times 1$. So $|Fix(e)| = q^{4n+1}$.

(2) $a$: $n \times 4$; $1 \times 1$. So $|Fix(a)| = q^{n+1}$.

(3) $b$: $2n \times 2$; $1 \times 1$. So $|Fix(b)| = q^{2n+1}$.

(4) $c$: $n \times 4$; $1 \times 1$. So $|Fix(c)| = q^{n+1}$.

(5) $p$: $n \times 2$; $(2n+1) \times 1$. So $|Fix(p)| = q^{3n+1}$.

(6) $q$: $n \times 2$; $(2n+1) \times 1$. So $|Fix(q)| = q^{3n+1}$.

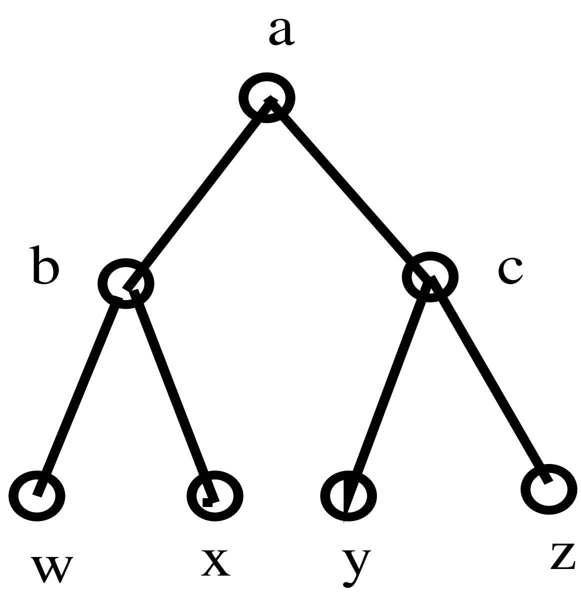
(7) $r$: $2n \times 2$; $1 \times 1$. So $|Fix(r)| = q^{2n+1}$.

(8) $s$: $2n \times 2$; $1 \times 1$. So $|Fix(s)| = q^{2n+1}$.

So the mumber of colorings is

$$\frac{1}{8}(q^{4n+1} + 2 \cdot q^{3n+1} + 3 \cdot q^{2n+1} + 2 \cdot q^{n+1}).$$

2. How many ways are there of coloring the edges of the diagram below in $q$ colors if the group acting consists of sequences of rotations of the sub-tree under a vertex. Think of each permutation as a permutation of the letters $x, y, z, w$. First determine the group elements. There are 8 in all.




**Solution:** The group can be expressed as

$$G = \{e, a, b, c, a \circ b, a \circ c, b \circ c, a \circ b \circ c\}.$$

To verify this, one first checks that there actions on $x, y, z, w$ are all different: The number at the right hand end is the number of cycles in the permutation.

| | | | | | |
|---|---|---|---|---|---|
| $e$ | $w$ | $x$ | $y$ | $z$ | $6 \times 1$ |
| $a$ | $z$ | $y$ | $x$ | $w$ | $3 \times 2$ |
| $b$ | $x$ | $w$ | $y$ | $z$ | $5 = 4 \times 1 + 1 \times 2$ |
| $c$ | $w$ | $x$ | $z$ | $y$ | $5 = 4 \times 1 + 1 \times 2$ |
| $a \circ b$ | $z$ | $y$ | $w$ | $x$ | $2 = 1 \times 2 + 1 \times 4$ |
| $a \circ c$ | $y$ | $z$ | $x$ | $w$ | $2 = 1 \times 2 + 1 \times 4$ |
| $b \circ c$ | $x$ | $w$ | $z$ | $y$ | $4 = 2 \times 1 + 2 \times 2$ |
| $a \circ b \circ c$ | $y$ | $z$ | $w$ | $x$ | $3 = 3 \times 2$ |

After this one verifies that all other combinations yield members of $G$. This follows from the identities:

$$a \circ a = b \circ b = c \circ c = e; b \circ a = a \circ c; c \circ a = a \circ b; c \circ b = b \circ c.$$

It follows that the number of colorings is

$$\frac{1}{8}(q^6 + q^3 + q^5 + q^5 + q^2 + q^2 + q^4 + q^3).$$

3. How many ways are there of placing $k$ 1's on a regular convex $n$-gon if each 1 must be seperated by at least 2 0's. You can assume that the group acting consists of the rotations through $\frac{2\pi j}{n}$ for $j = 0, 1, \ldots, n-1$. To simplify matters, you can assume that $k$ is prime.

It will help to understand the material on pp20/21 of the notes: "Polya theory of counting".

**Solution:** The rotation $e_j, j \geq 1$ through $\frac{2\pi j}{n}$ breaks up into $d$ cycles of length $\frac{n}{d}$ where $d$ is the g.c.d. of $j, n$. An assignment of 0's and 1's will be in $Fix(e_j)$ iff the cycles are all 1's or all 0's. Because $k$ is prime, if $Fix(e_j) \neq \emptyset$ then there must be one cycle of 1's and $d-1$ of 0's. This means that $\frac{n}{d} = k$ and hence $k$ divides $n$. In which case, if $j \geq 1$ then

$$|Fix(e_j)| = \begin{cases} d & j = d, 2d, \ldots, (k-1)d \\ 0 & otherwise \end{cases}$$

So, the number of ways is

$$\begin{cases} \frac{1}{n}\frac{n}{k}\binom{n-2k-1}{k-1} = \frac{1}{k}\binom{n-2k-1}{k-1} & k \text{ does not divide } n \\ \frac{1}{n}\left(\frac{n}{k}\binom{n-2k-1}{k-1} + (k-1)d\right) = \frac{1}{k}\left(\binom{n-2k-1}{k-1} + k - 1\right) & n = kd. \end{cases}$$