21-301 Combinatorics
Homework 3
Due: Monday, October 30

1. Suppose that in the Tower of Hanoi problem there are n rings and 4 pegs. Let H_n denote the minimum number of moves required to move the rings to peg 4. Show that

$$H_n \leq 2H_{n-m} + 2^m - 1$$

holds for any $1 \leq m \leq n$.

Suppose next that we take $m = \lceil n^{1/2} \rceil$. Use this to prove that

$$H_n \leq 2^{3n^{1/2}}. \tag{1}$$

The inequality $(1-x)^{1/2} \leq 1 - x/2$ for $0 \leq x \leq 1$ could be useful. You can assume that (1) is true for $n \leq 10$.

Solution One can proceed as follows: Move the top $n-m$ rings on peg 1 onto peg 2 in H_{n-m} moves, using all pegs. Then move the remaining m rings on peg 1 onto peg 4 in $2^m - 1$ moves, using pegs 1,3,4. Then move the $n-m$ rings on peg 2 onto peg 4 in H_{n-m} moves, using all pegs. The number of moves used is $2H_{n-m} + 2^m - 1$.

Now assume that $m = \lceil n^{1/2} \rceil$ and also assume inductively that $H_{n-m} \leq 2^{3(n-m)^{1/2}}$. Then equation (1) is true for small n, say $n \leq 10$ and for larger n we have

$$H_n \leq 2^{3(n-m)^{1/2}+1} + 2^m - 1 = 2^{3n^{1/2}(1-m/n)^{1/2}+1} + 2^m - 1 \leq$$
$$2^{3n^{1/2}(1-m/(2n))+1} + 2^m - 1 \leq 2^{3n^{1/2}-1/2} + 2^m - 1 \leq 2^{3n^{1/2}}.$$

2. Show that the number of sequences out of $\{a,b,c\}^n$ which do not contain a consecutive sub-sequence of the form abc satisfies the recurrence $b_0 = 1, b_1 = 3, b_2 = 9$ and

$$b_n = 2b_{n-1} + c_n \tag{2}$$
$$c_n = c_{n-1} + b_{n-2} + c_{n-2} + b_{n-3} \tag{3}$$

where c_n is the number of such sequences that start with a.

Now find a recurrence only involving b_n, by using (2) to eliminate c_n from (3).

Solution: There are $2b_{n-1}$ sequences of the required form that start with b or c. There are c_n sequences that start with a. This explains (2).

There are c_{n-1} sequences that start with aa, b_{n-2} sequences that start with ac, c_{n-2} sequences that start with aba and b_{n-3} sequences that start with abb. This covers the possibilities for sequences starting with a.

We have

$$b_n - 2b_{n-1} = b_{n-1} - 2b_{n-2} + b_{n-2} + b_{n-2} - 2b_{n-3} + b_{n-3}$$

and so

$$b_n = 3b_{n-1} - b_{n-3}.$$

3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the recurrence relation

$$a_n + 4a_{n-1} + 3a_{n-2} = 2n + 1 \quad \text{for } n \geq 2$$

with initial conditions $a_0 = 1$ and $a_1 = 4$. Determine the generating function for this sequence, and use the generating function to determine a_n for all n.

Solution:

$$\begin{aligned}
\sum_{n=2}^{\infty}(a_n + 4a_{n-1} + 3a_{n-2})x^n &= \sum_{n=2}^{\infty}(2n+1)x^n \\
a(x) - 1 - 4x + 4x(a(x) - 1) + 3x^2 a(x) &= \frac{1+x}{(1-x)^2} - 1 - 3x \\
a(x)(1 + 4x + 3x^2) &= \frac{1+x}{(1-x)^2} + 5x
\end{aligned}$$

$$\begin{aligned}
a(x) &= \frac{1}{(1+3x)(1-x)^2} + \frac{5x}{(1+x)(1+3x)} \\
&= \frac{5/2}{1+x} + \frac{-31/16}{1+3x} + \frac{3/16}{1-x} + \frac{1/4}{(1-x)^2} \\
&= \sum_{n=0}^{\infty}\left(\frac{5}{2}(-1)^n - \frac{31}{16}(-3)^n + \frac{3}{16} + \frac{1}{4}(n+1)\right)x^n.
\end{aligned}$$

So

$$a_n = \frac{5}{2}(-1)^n - \frac{31}{16}(-3)^n + \frac{3}{16} + \frac{1}{4}(n+1) \qquad \text{for } n \geq 0.$$