21-301 Combinatorics
Homework 3
Due: Monday, October 3

1. Suppose that in the Tower of Hanoi problem there are n rings and 4 pegs. Let H_n denote the minimum number of moves required to move the rings to peg 4. Show that

$$H_n \leq 2H_{n-m} + 2^m - 1$$

holds for any $1 \leq m \leq n$.

Suppose next that we take $m = \lceil n^{1/2} \rceil$. Use this to prove that

$$H_n \leq 2^{3n^{1/2}}. \tag{1}$$

The inequality $(1-x)^{1/2} \leq 1 - x/2$ for $0 \leq x \leq 1$ could be useful. You can assume that (1) is true for $n \leq 10$.

2. Show that the number of sequences out of $\{a, b, c\}^n$ which do not contain a consecutive sub-sequence of the form abc satisfies the recurrence $b_0 = 1, b_1 = 3, b_2 = 9$ and

$$b_n = 2b_{n-1} + c_n \tag{2}$$
$$c_n = c_{n-1} + b_{n-2} + c_{n-2} + b_{n-3} \tag{3}$$

where c_n is the number of such sequences that start with a.

Now find a recurrence only involving b_n, by using (2) to eliminate c_n from (3).

3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the recurrence relation

$$a_n + 4a_{n-1} + 3a_{n-2} = 2n + 1 \quad \text{for } n \geq 2$$

with initial conditions $a_0 = 1$ and $a_1 = 4$. Determine the generating function for this sequence, and use the generating function to determine a_n for all n.