21-301 Combinatorics
Homework 2
Due: Friday, September 18

1. Prove that for any $k, n \geq 1$ that

$$\sum_{\substack{a_1+\cdots+a_{2k}=n \\ a_1,\ldots,a_{2k}\geq 0}} 3^{a_1+\cdots+a_k}(-1)^{a_{k+1}+\cdots+a_{2k}}\binom{n}{a_1,\ldots,a_{2k}} = (2k)^n.$$

Solution: We start with the expression

$$(x_1+x_2+\cdots+x_{2k})^n = \sum_{\substack{a_1+\cdots+a_{2k}=n \\ a_1,\ldots,a_{2k}\geq 0}} \binom{n}{a_1,\ldots,a_{2k}} x_1^{a_1}x_2^{a_2}\cdots x_{2k}^{a_{2k}}$$

and then put $x_1 = x_2 = \cdots = x_k = 3$ and $x_{k+1} = x_{k+2} = \cdots = x_{2k} = -1$ to get the result we want.

2. **(a)** Let $\mathcal{S}_{k,\ell,m}$ denote the collection of k-sets $\{1 \leq i_1 < i_2 < \cdots < i_k \leq m-\ell+1\} \subseteq [m]$ such that $i_{t+1} - i_t \geq \ell$ for $1 \leq t < k$. Show that

$$|\mathcal{S}_{k,\ell,m}| = \binom{m-(\ell-1)k}{k}.$$

(b) How many of the ℓ^n sequences $x_1x_2\cdots x_n$, $x_i \in \{a_1, a_2, \ldots, a_\ell\}$, $i = 1, 2, \ldots, n$ are there such that $a_1a_2\cdots a_\ell$ does not appear as a consecutive subsequence e.g. if $n = 6$ and $\ell = 5$ then we include $a_1a_4a_2a_2a_3a_5$ in the count, but we exclude $a_1a_2a_3a_4a_5a_1$.

[You should use Inclusion-Exclusion and expect to have your answer as a sum.]

Solution:

(a) For a first argument, let $z_1 = i_1, z_2 = i_2 - i_1, \ldots, z_k = i_k - i_{k-1}, z_{k+1} = m - i_k$. We can count the number of choices for $z_1, z_2, \ldots, z_{k+1}$. But these are the solutions to

$$z_1 + z_2 + \cdots + z_{k+1} = m,\ z_1 \geq 1, z_2, z_3, \ldots, z_k \geq \ell, z_{k+1} \geq \ell - 1.$$

The number of such is

$$\binom{m-1-\ell(k-1)-(\ell-1)+k+1-1}{k+1-1} = \binom{m-(\ell-1)k}{k}.$$

Alternatively, we can represent a k-set by a sequence of k 1's and $m-k$ 0's in the usual way. Now we need every pair of 1's separated by at least $\ell-1$ 0's. We can start with a sequence of $m-(\ell-1)k$ 0's, choose k of them and replace each of these k 0's by $100\cdots00$ ($\ell-1$ 0's here). This process is reversible. For the 0,1 sequences we are counting each 1 is followed by at least $\ell-1$ 0's. Just replace $100\cdots00$ by 0 to get a sequence of $m-k$ 0's.

(b) Let $A = \{a_1, a_2, \ldots, a_\ell\}^n$. Then let

$$A_k = \{x \in A:\ x_{k+i-1} = a_i, 1 \leq i \leq \ell\}$$

for $k = 1, 2, \ldots, n - \ell + 1$.

Then

$$|A_S| = \begin{cases} \ell^{n-\ell|S|} & S \in \mathcal{S}_{|S|,\ell,m} \\ 0 & S \notin \mathcal{S}_{|S|,\ell,m} \end{cases}.$$

Then we must compute

$$\begin{aligned} \left|\bigcap_{i=1}^{m} \bar{A}_i\right| &= \sum_{S \in \mathcal{S}} (-1)^{|S|} |A_S| \\ &= \sum_{S \in \mathcal{S}_{|S|,\ell,m}} (-1)^{|S|} \ell^{n-\ell|S|} \\ &= \ell^n \sum_{k=0}^{m} (-1)^k |\mathcal{S}_{k,\ell,m}| \ell^{-\ell k} \\ &= \ell^n \sum_{k=0}^{m} (-1)^k \binom{m-(\ell-1)k}{k} \ell^{-\ell k} \end{aligned}$$

3. How many ways are there of placing m distinguishable balls into n boxes so that no box contains more than B balls.
   (You should use Inclusion-Exclusion and expect to have your answer as a sum.)

   Solution: Let A_i be the set of allocations in which box i contains more than B balls. Then

$$|A_1| = \sum_{k=B+1}^{m} \binom{m}{k} (n-1)^{m-k}$$

and

$$|A_{\{1,2\}}| = \sum_{k,\ell=B+1}^{m} \binom{m}{k, \ell, m-k-\ell} (n-2)^{m-k-\ell}.$$

In general, if $|S| = s$ then

$$|A_S| = a_s = \sum_{k_1,k_2,\ldots,k_s=B+1}^{m} \binom{m}{k_1, k_2, \ldots, k_s, m-k_1-k_2-\cdots-k_s} (n-s)^{m-k_1-k_2-\cdots-k_s}.$$

Thus, the number of allocations in which no box gets more than B balls is

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} a_s.$$