21-301 Combinatorics
Homework 2
Due: Friday, September 16

1. Prove that for any $k, n \geq 1$ that

$$\sum_{\substack{a_1+\cdots+a_{2k}=n \\ a_1,\ldots,a_{2k}\geq 0}} 3^{a_1+\cdots+a_k}(-1)^{a_{k+1}+\cdots+a_{2k}}\binom{n}{a_1,\ldots,a_{2k}} = (2k)^n.$$

2. **(a)** Let $\mathcal{S}_{k,\ell,m}$ denote the collection of k-sets $\{1 \leq i_1 < i_2 < \cdots < i_k \leq m-\ell+1\} \subseteq [m]$ such that $i_{t+1} - i_t \geq \ell$ for $1 \leq t < k$. Show that

$$|\mathcal{S}_{k,\ell,m}| = \binom{m-(\ell-1)k}{k}.$$

   (b) How many of the ℓ^n sequences $x_1x_2\cdots x_n$, $x_i \in \{a_1, a_2, \ldots, a_\ell\}$, $i = 1, 2, \ldots, n$ are there such that $a_1a_2\cdots a_\ell$ does not appear as a consecutive subsequence e.g. if $n = 6$ and $\ell = 5$ then we include $a_1a_4a_2a_2a_3a_5$ in the count, but we exclude $a_1a_2a_3a_4a_5a_1$.

   [You should use Inclusion-Exclusion and expect to have your answer as a sum.]

3. How many ways are there of placing m distinguishable balls into n boxes so that no box contains more than B balls.
   (You should use Inclusion-Exclusion and expect to have your answer as a sum.)