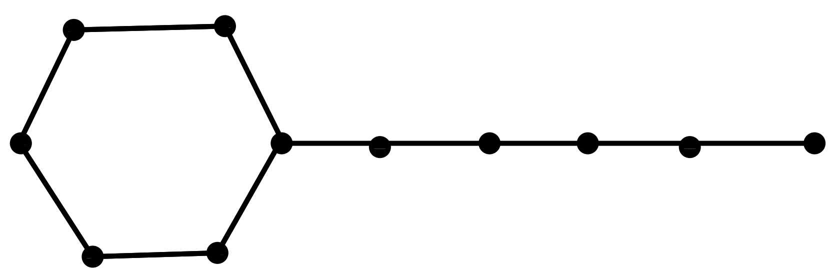
21-301 Combinatorics
Homework 1
Due: Wednesday, September 7

1. How many integral solutions of
$$x_1 + x_2 + x_3 + x_4 + x_5 = 100$$
satisfy $x_1 \geq 5$, $x_2 \geq 8$, $x_3 \geq -2$, $x_4 \geq 3$ and $x_5 \geq 1$?
Solution Let
$$y_1 = x_1 - 5, \quad y_2 = x_2 - 8, \quad y_3 = x_3 + 2, \quad y_4 = x_4 - 3, \quad y_5 = x_5 - 1.$$
An integral solution of $x_1 + x_2 + x_3 + x_4 + x_5 = 100$ such that $x_1 \geq 5$, $x_2 \geq 8$, $x_3 \geq -2$, $x_4 \geq 3$ and $x_5 \geq 1$ corresponds to an integral solution of $y_1 + y_2 + y_3 + y_4 + y_5 = 85$ such that $y_1, \ldots, y_5 \geq 0$. From a result in class,
$$|\{(y_1, y_2, y_3, y_4, y_5) : y_1, \ldots, y_5 \in Z_+ \text{ and } y_1 + \cdots + y_5 = 85\}| = \binom{85+5-1}{5-1} = \binom{89}{4}.$$

2. Show that
$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{2} = \binom{n}{2} 2^{n-2}.$$
The LHS is the number of choices of pairs (A, B) where $A, B \subseteq [n]$ and $A \subseteq B$ and $|A| = 2$. The RHS is the number of choices of pairs (A, C) where $A \subseteq [n]$ and $C \subseteq [n] \setminus A$. The map $(A, B) \to (A, B \setminus A)$ is a bijection that shows the two sides are equal.

3. How many ways are there of placing k 1's and $n - k$ 0's at the vertices of the cycle and at the vertices of the path in the diagram below so that each 1 is separated by at least one 0? Thus there will either be $2k$ 1's altogether, when the common vertex has a 0 on it, or $2k - 1$ 1's altogether, when the common vertex has a 1 on it.



The cycle and the path both have n vertices. There are $2n - 1$ vertices altogether.

Solution We split the arrangements of 0's and 1's into two sets S_0, S_1. S_1 will be the set of arrangements where there is a 1 at the intersection of the polygon and the path.

For S_0, we place a 0 at the intersection. Focus first on the polygon and consider the sequence $a_0, a_1, \ldots a_k$ describing the number of 0's we encounter between 1's as we traverse one polygon in a clockwise manner. Here we have $a_0 + a_1 + \cdots + a_k = n - k - 1$ and

$a_0, a_k \geq 0$ and $a_i \geq 1$ for $i \neq 0, k$. The number of choices for $a_0, a_1, \ldots a_k$ is therefore $\binom{(n-k-1)+(k+1)-1-(k-1)}{(k+1)-1} = \binom{n-k}{k}$. The number of ways of putting the k 1's on the path is the number of choices of $a_1 \geq 0, a_2, \ldots, a_{k-1} \geq 1, a_{k+1} \geq 0$ such that $a_1 + \cdots + a_{k+1} = n-k$ and this is $\binom{n-k}{k}$. So $|S_0| = \binom{n-k}{k}^2$

In class we showed that if we put a 1 at a particular point of one polygon, then the number of ways of putting $k-1$ 1's onto the rest of the cycle is $\binom{n-k-1}{k-1}$. The number of ways of putting the $k-1$ 1's on the path is the number of choices of $a_1, \ldots, a_{k-1} \geq 1, a_k \geq 0$ such that $a_1 + \cdots + a_k = n-k$ and this is $\binom{n-k}{k-1}$. This implies that $|S_1| = \binom{n-k-1}{k-1}\binom{n-k}{k-1}$, since we choose an arrangement for the polygon and an arrangement for the path.

The total is

$$\binom{n-k}{k}^2 + \binom{n-k-1}{k-1}\binom{n-k}{k-1}.$$