<u>9\9\11</u>

Principle of Inclusion-Exclusion

(PIE)

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$A_1, A_2 \subseteq A$, $\quad \bar{A}_1 = A \setminus A_1$, $\bar{A}_2 = A \setminus A_2$

$$|\bar{A}_1 \cap \bar{A}_2| = |\overline{A_1 \cup A_2}|$$

$$= |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| =$$

$$|A|$$

$$- |A_1| - |A_2| - |A_3|$$

$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$$

$$- |A_1 \cap A_2 \cap A_3|$$

$A_1, A_2, \ldots, A_N \subseteq A$

For $S \subseteq [N]$,

$\boxed{\text{Not } S \subseteq A}$

$A_S = \bigcap_{i \in S} A_i$

$A_{\{2,3,8\}} = A_2 \cap A_3 \cap A_8$

$A_\emptyset = A$

$$\left| \bigcap_{i=1}^{N} \overline{A}_i \right| = \sum_{S \subseteq [N]} (-1)^{|S|} |A_S|$$

How many integers are there in in [1000] that are not divisible by 4, 7, 12, 15

$A = [1000]$

What I want is

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4|$$

$A_1 = \{i : 4|i\}$

$A_2 = \{i : 7|i\}$

$A_3 = \{i : 12|i\}$

$A_4 = \{i : 15|i\}$

$$= |A| - |A_1| - |A_2| - |A_3| - |A_4| + |A_{\{1,2\}}| + \cdots + |A_{\{1,2,3,4\}}|$$

$|A| = 1000$

$|A_1| = 250$; $|A_2| = 142$; $|A_3| = 83$, $|A_4| = 66$

$|A_{\{1,2\}}| = 35$; $|A_{\{1,3\}}| = 83$; $|A_{\{1,4\}}| = 16$

$|A_{\{2,3\}}| = 11$; $|A_{\{2,4\}}| = 9$, $|A_{\{3,4\}}| = 16$

$|A_{\{1,2,3\}}| = 11$; $|A_{\{1,2,4\}}| = 2$; $|A_{\{1,3,4\}}| = 16$

$|A_{\{2,3,4\}}| = 2$; $|A_{\{1,2,3,4\}}| = 2$

Total $= 1000 - 250 - 142 - 83 - 66 + 35 +$

Derangements

A derangement of $[n]$ is a permutation π s.t. $\pi(i) \neq i, \forall i$

$A = \{\text{all permutations}\}$

what we want is

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_i} \cdots \cap \overline{A_n}|$$

$A_1 = \{\pi : \pi(1) = 1\}$

$A_i = \{\pi : \pi(i) = i\}$

$\#$ derangements $= \displaystyle\sum_{S \subseteq [n]} (-1)^{|S|} |A_S|$

$|A_\emptyset| = n!$

$|A_{\{1\}}| = (n-1)!$

$\underline{1}\ \times\ \times\ \times\ \times\ \times\ \times\ \times\ \times$

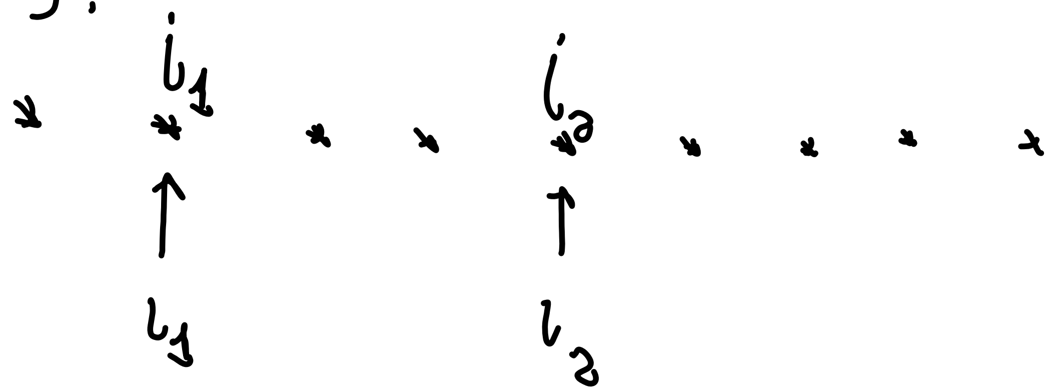
$|A_{\{1,2\}}| = (n-2)!$

$\underline{1}\ \ 2\ \times\ \times\ \times\ \times\ \times\ \times$

$A_1 \cap A_2$

$\pi(1) = 1,\ \pi(2) = 2$

$$|A_S| = (n-|S|)!$$



$S = i_1, i_2, \ldots i_s$

$$\# \text{ derangements} = \sum_{S \subseteq [n]} (-1)^{|S|} (n-|S|)!$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)!$$

$(-1)^k \frac{n!}{k!(n-k)!} \times (n-k)!$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \sim n! \times e^{-1} \text{ for large } n$$

$1 - 1 + \frac{1}{2!} - \frac{1}{3!} \ldots$