9/2/11

Identities involving binomial coefficients:

(1) $\binom{n}{r} = \binom{n}{n-r}$ Obvious from formula

choose r elements to include

choose $n-r$ elements to exclude

"Pascal Triangle"

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

$\#$ that have n+1 in them

$\#$ that do not have n+1 in them

$\#$ (k+1)-subsets of [n+1]

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

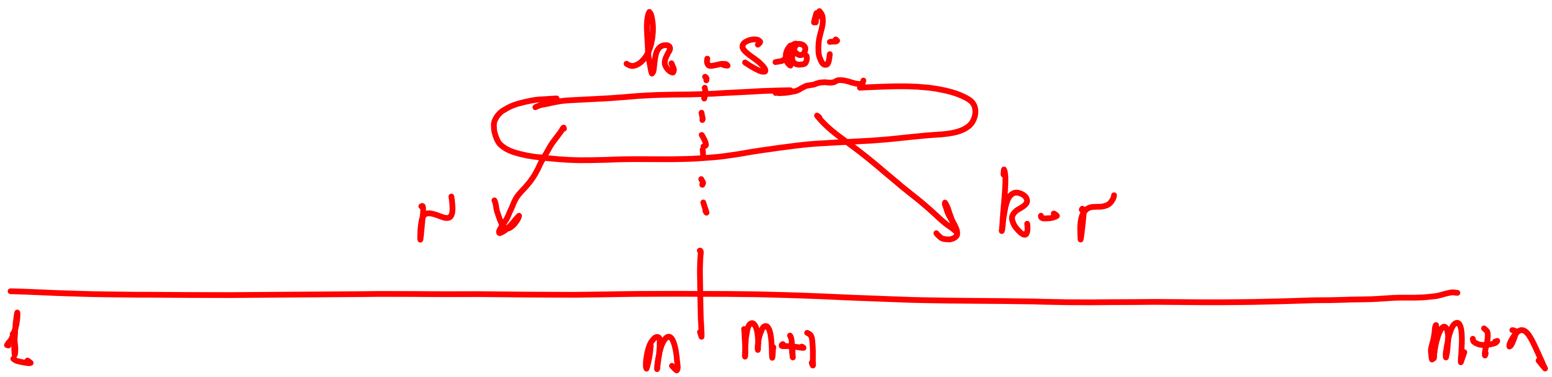
$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots \binom{m}{k} + \cdots \binom{n}{k} = \binom{n+1}{k+1}$$

# of subsets that contain $m+1$ as their largest element

$(k+1)$-subsets of $[n+1]$

Vandermonde's Identity

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}$$



Binomial Theorem

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r \qquad \underline{\forall x}$$

$(1+x)(1+x)(1+x) \cdots \cdots (1+x)$

Coefficient of x^r: when multiplying out choose 1 or x from each bracket.

Choose x r times contributes 1 to coefficient of x^r.

choices is $\binom{n}{r}$

$\underline{x=1}$ $\qquad \binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=(1+1)^n=2^n$

$\underline{x=-1}$ $\qquad \binom{n}{0}-\binom{n}{1}+\cdots+(-1)^n\binom{n}{n}=(1-1)^n=0$

$$\binom{n}{0}+\binom{n}{2}+\cdots=\binom{n}{1}+\binom{n}{3}+\cdots$$

even sets = # odd sets

Differentiate

$$\sum_{k=0}^{n} k \binom{n}{k} x^{k-1} = n(1+x)^{n-1}$$

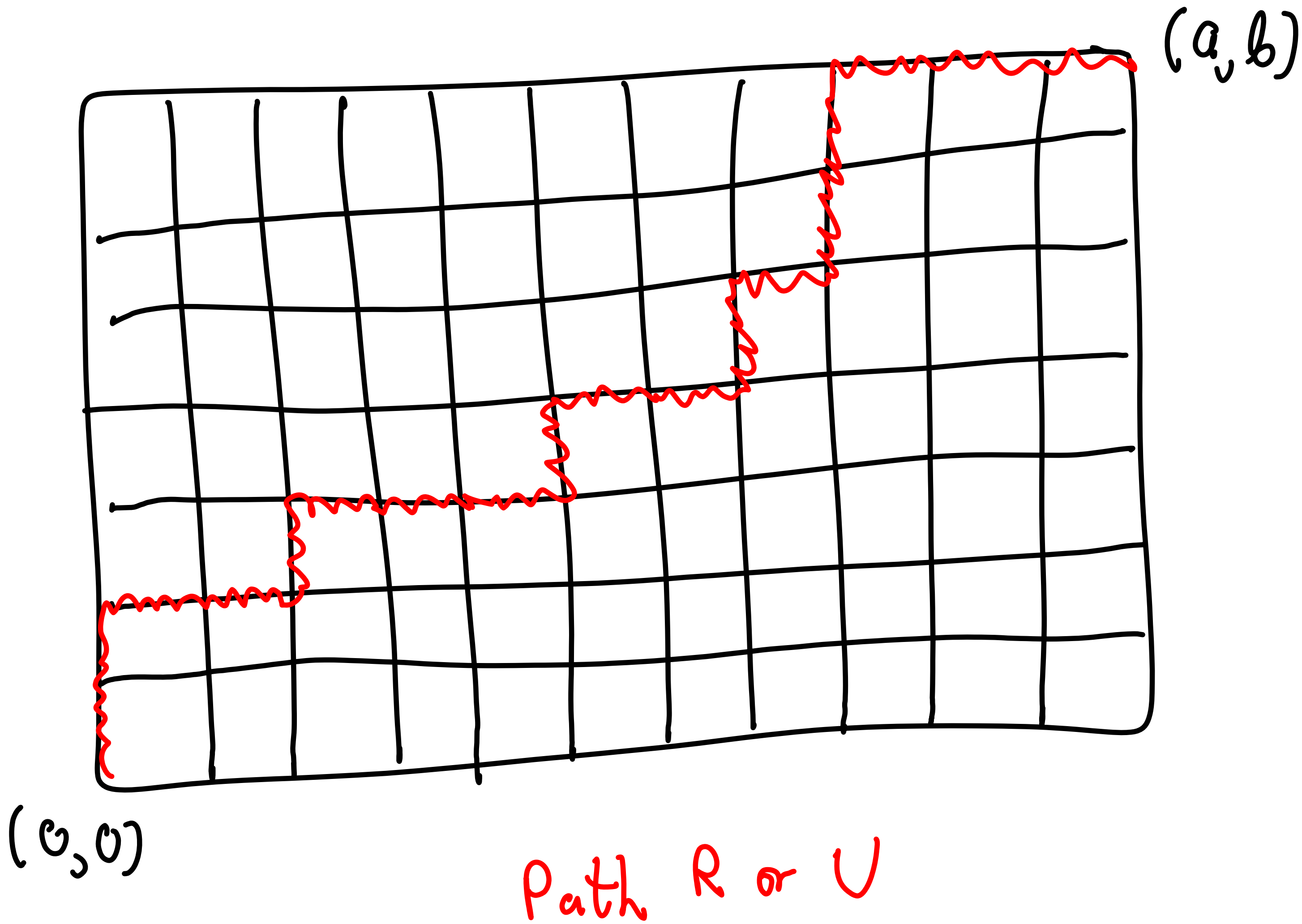
Put $x=1$

$$\sum_{k=0}^{n} k \binom{n}{k} = n\,2^{n-1}$$

Choose set and then one element from set

choose an element x and a set not containing x.

Grid Path Problems



Questions:

1) How many paths from $(0,0)$ to (a,b)?

$a < b$

2) How many stay above the diagonal (except possibly at the ends)

diagonal $= (0,0), (1,1), (2,2), \ldots$

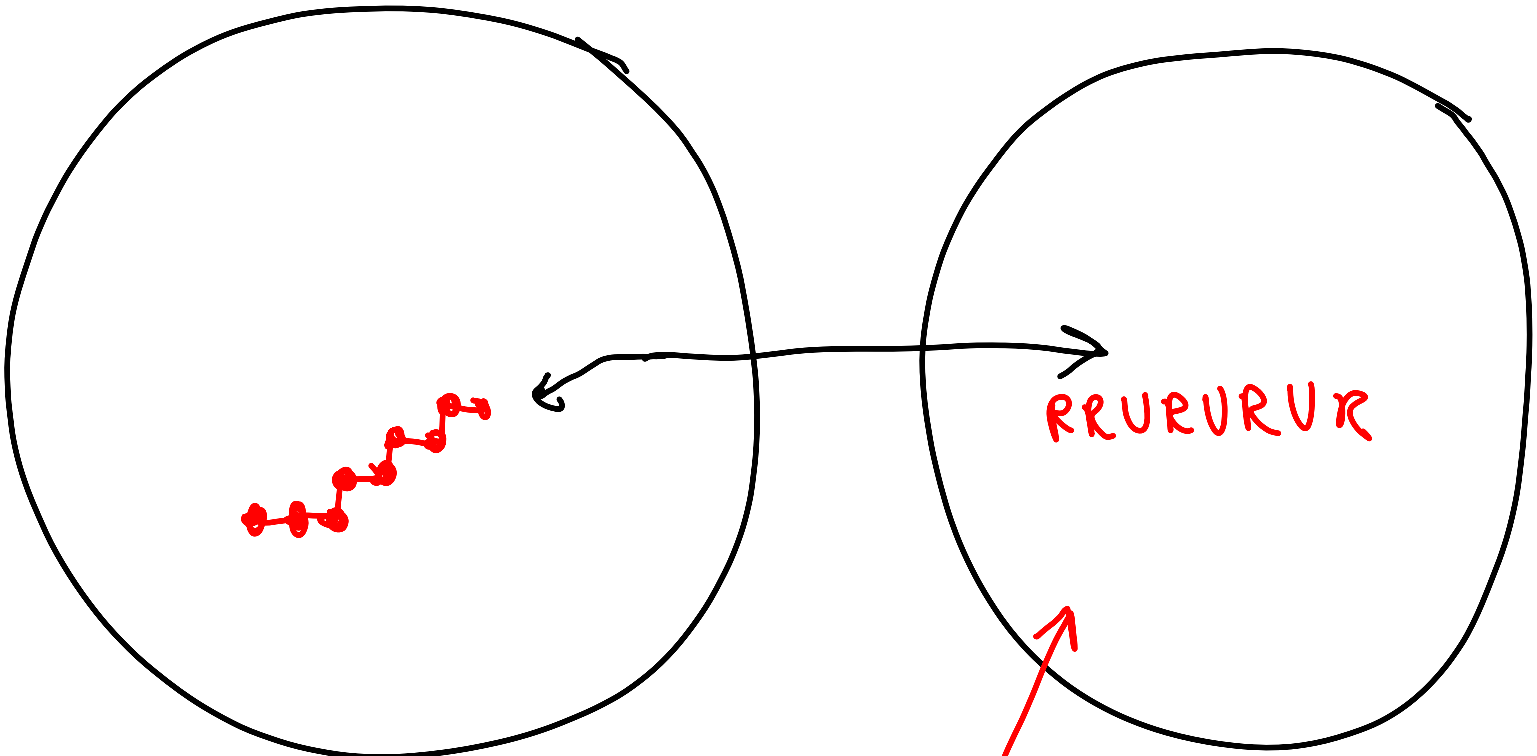
$a \leq b$

3) How many never go below the diagonal?

* Ballot Problem: $a = b$
Two candidates for election get same # of votes.
How many ways in count for first to not be behind.

PATHS(a,b) :

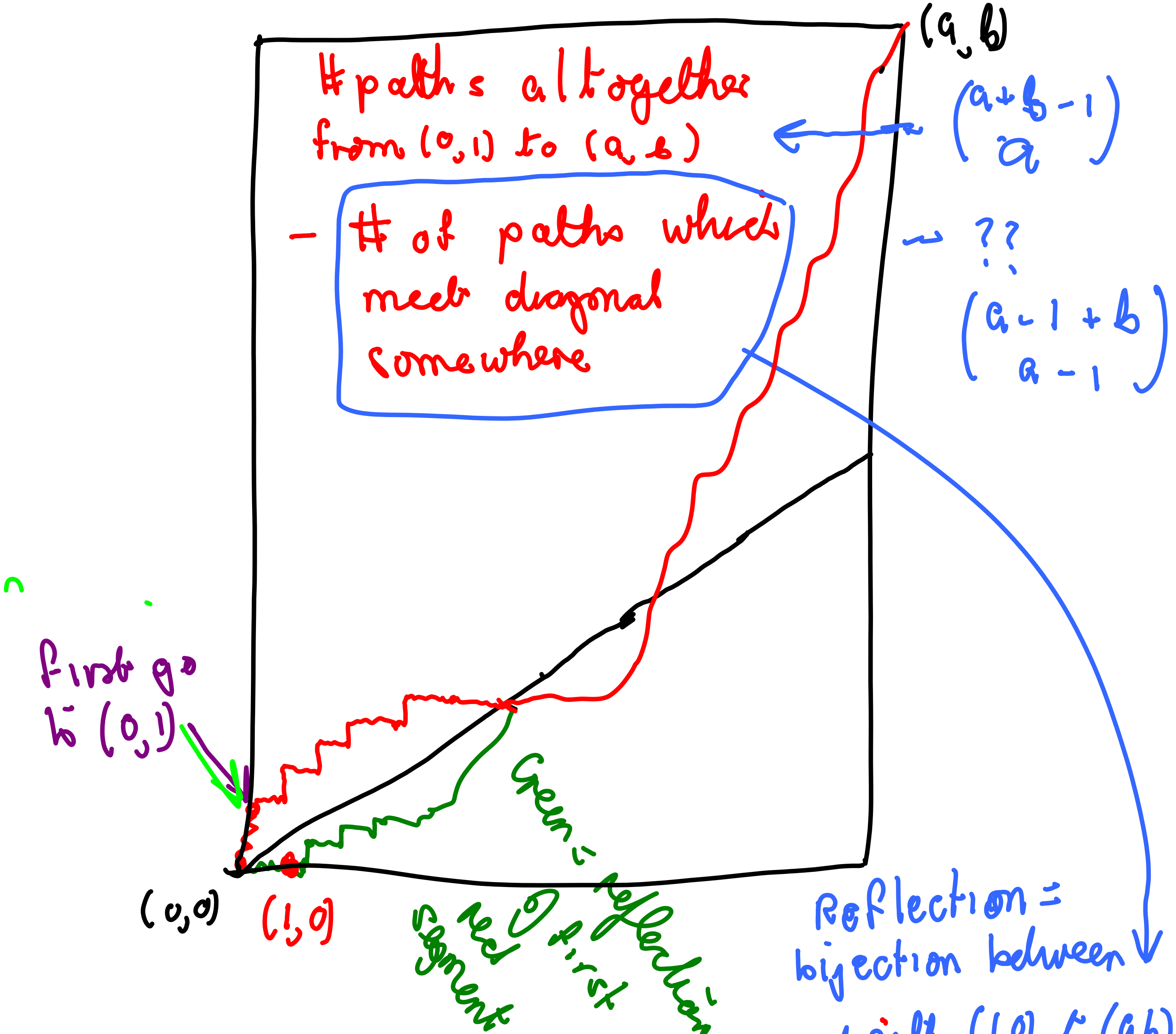
$\{ \sigma \in \{R, U\}^{a+b} :$
with $a \times R$ e $b \times U$



#elements $= \binom{a+b}{a}$

$a+b \Rightarrow$ . . . .

choose a places for an R



(a, b)
paths altogether from (0,1) to (a,b)
$\binom{a+b-1}{a}$
− # of paths which meet diagonal somewhere
??
$\binom{a-1+b}{a-1}$
first go to (0,1)
(0,0)
(1,0)
Green = reflection of first red segment
Reflection = bijection between and paths (1,0) to (a,b)