9\23\11

Solution of linear recurrences.

$$a_n - 4a_{n-1} + 4a_{n-2} = 0 \qquad n \geq 2$$

$a_0 = a_1 = 1$

Equation implies

$a_2 = 0$, $a_3 = -4$, ...

Transform into an equation for $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

Take each equation, multiply by x^n and sum.

$$\sum_{n\geq 2} (a_n - 4a_{n-1} + 4a_{n-2})x^n = 0$$

$$\sum_{n\geq 2} a_n x^n - 4\sum_{n\geq 2} a_{n-1}x^n + 4\sum_{n\geq 2} a_{n-2}x^n = 0$$

$$a(x) - a_0 - a_1 x$$
$$= a(x) - 1 - x$$

$$a_1x^2 + a_2x^3 + \dots$$
$$= x(a_1x + a_2x^2 + \dots)$$
$$= x(a(x) - a_0)$$
$$= x(a(x) - 1)$$

$$a_0x^2 + a_1x^3 + \dots$$
$$= x^2 a(x)$$

$$(a(x) - 1 - x) - 4x(a(x) - 1) + 4x^2 a(x) = 0$$

$$a(x)(1 - 4x + 4x^2) = 1 + x - 4x$$
$$= 1 - 3x$$

$$a(x) = \frac{1 - 3x}{1 - 4x + 4x^2}$$

$$a_n = [x^n]\left(\quad \downarrow \quad \right)$$

$$\frac{1 - 3x}{1 - 4x + 4x^2} = \frac{1 - 3x}{(1 - 2x)^2} = \frac{A}{1 - 2x} + \frac{B}{(1 - 2x)^2}$$

$$\frac{1-3x}{(1-2x)^2} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2}$$

$$1-3x = A(1-2x) + B \qquad \forall x$$

$$A + B = 1$$

$$A = 3/2$$

$$a_n = 3\cdot 2^{n-1} - (n+1)2^{n-1}$$
$$= -(n-2)2^{n-1}$$

$$a(x) = \frac{3/2}{1-2x} - \frac{1/2}{(1-2x)^2}$$

$$= \frac{3}{2}\sum_{n=0}^{\infty} 2^n x^n - \frac{1}{2}\sum_{n=0}^{\infty} (n+1)2^n x^n$$

$$a_n - 4a_{n-1} + 4a_{n-2} = n \qquad n \geq 2$$

$$a_0 = a_1 = 1$$

$$\sum_{n=2}^{\infty} (a_n - 4a_{n-1} + 4a_{n-2}) x^n = \sum_{n \geq 2}^{\infty} n x^n \;*$$

$$a(x)(1-2x)^2 - (1-3x) = \frac{x}{(1-x)^2} - x$$

$$a(x) = \frac{x}{(1-x)^2(1-2x)^2} + \frac{1-4x}{(1-2x)^2}$$

$$* \quad 2x^2 + 3x^3 + 4x^4 + \cdots = \frac{x}{(1-x)^2} - x$$

$$Q(x) = \frac{x}{(1-x)^2(1-2x)^2} + \frac{1-4x}{(1-2x)^2}$$

$$= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-2x} + \frac{D}{(1-2x)^2}$$

Product of generating functions

$$a(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad b(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$a(x)\, b(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Derangements

$d_n = \#$ of derangements.

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}$$

choose k indices I such that $\pi(i) = i$, $i \in I$

and $\pi(i) \neq i$ for $i \notin I$

$$\begin{array}{c} i \\ \downarrow \\ i \in I \end{array}$$

$\pi(k)$ × · × × ↓ · × × · ·