9\21\11

Recurrence Relations

Think of

$$a_0, a_1, \ldots a_n, \ldots$$

as an unknown sequence.

Think of the a_n's as unknowns.

A recurrence relation

$$a_n = f_n(a_{n-1}, a_{n-2}, \ldots, a_0)$$

is a set of equations in these unknowns.

Idea

1) Unknown Sequence

2) Construct recurrence satisfied by the sequence

3) Solve the recurrence.

Linear Recurrence

Fibonacci Sequence

$$a_n = a_{n-1} + a_{n-2} \quad , \quad n \geq 2$$

$$a_0 = a_1 = 1$$

$1, 1, 2, 3, 5, 8, 13, \ldots$

$B_n = \{ x \in \{a, b, c\}^n :$ aa does not occur as a subsequence $\}$

$b_n = |B_n|$

$b_1 = 3$; $b_2 = 8$; $b_3 = 27 - \overset{aaa}{1} - 2 \times 2$
$= 22$

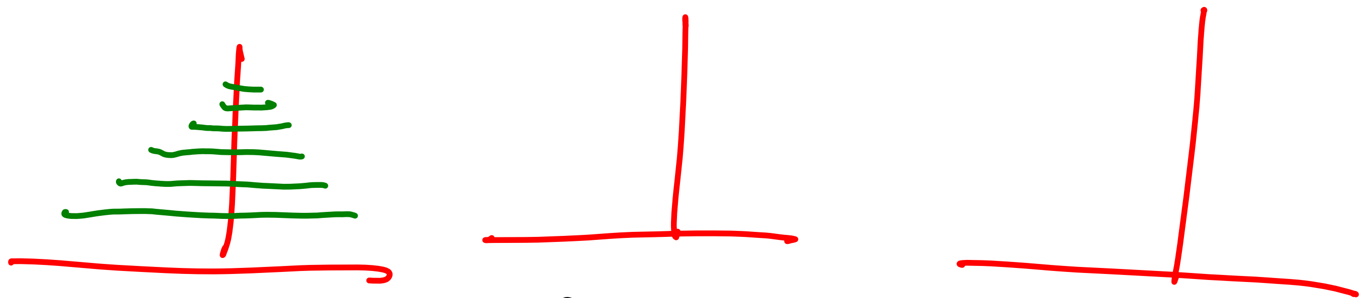
Claim: $b_n = 2 b_{n-1} + 2 b_{n-2}$

$$b_n = 2b_{n-1} + 2b_{n-2}$$

$$B_n = \left\{ \begin{array}{ll} b * * * * \cdots * & \leftarrow b_{n-1} \text{ of these} \\ c * * * * \cdots * & \leftarrow b_{n-1} \text{ of these} \\ a\ b * * * \cdots * & \leftarrow b_{n-2} \text{ of these} \\ a\ c * * * \cdots * & \leftarrow b_{n-2} \text{ of these} \end{array} \right.$$

Towers of Hanoi

$$H_n = 2H_{n-1} + 1 = 2^n - 1$$



Peg 1 Peg 2 Peg 3

Goal move rings from Peg 1 → Peg 3
but never place larger ring on top of smaller one

H_n = # moves (best possible)

$$= \underbrace{H_{n-1}}_{\text{move } n-1 \text{ to peg 2}} + \underbrace{1}_{\text{move largest ring}} + \underbrace{H_{n-1}}_{\text{move rest}}$$

Generating Functions

$$a_0, a_1, \dots \quad a_n, \dots$$

$$a(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

is the generating function for the sequence.

$$a_n = [x^n]\, a(x)$$

Unknown Sequence + Recurrence

$\Downarrow$

Unknown generating function + Equation

$\Downarrow$ Solve

Know generating function

$\Downarrow$

Known sequence.

Newton's Binomial Theorem

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

α is real (or even complex)

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

$$\frac{1}{(1-x)^m} = (1-x)^{-m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k$$

$$= \sum_{k=0}^{\infty} \binom{-m}{k} (-x)^k$$

$$\binom{-m}{k} = \frac{(-m)(-m-1)\cdots(-m-k+1)}{k!}$$

$$= (-1)^k \frac{m(m+1)\cdots(m+k-1)}{k!}$$

$$= (-1)^k \binom{m+k-1}{k}$$