9\12\11

Proof of P.I.E.

$A_1, A_2, \ldots, A_N$

$$\theta_{x,i} = \begin{cases} 1 & : x \in A_i \\ 0 & : \text{otherwise} \end{cases}$$

$$(1-\theta_{x,1})(1-\theta_{x,2})\ldots(1-\theta_{x,N}) = \begin{cases} 1 : x \in \bigcap_{i=1}^{N} \bar{A}_i \\ 0 : \text{otherwise} \end{cases}$$

$$\left|\bigcap_{i=1}^{N} \bar{A}_i\right| = \sum_{x \in A} (1-\theta_{x,1})(1-\theta_{x,2})\ldots(1-\theta_{x,N})$$

where you take $\theta_{x,i}$

$$= \sum_{x \in A} \sum_{S \subseteq [N]} (-1)^{|S|} \prod_{i \in S} \theta_{x,i}$$

$$= \sum_{S \subseteq [N]} (-1)^{|S|} \underbrace{\sum_{x \in A} \underbrace{\prod_{i \in S} \theta_{x,i}}_{= 1 \text{ iff } x \in \bigcap_{i \in S} A_i}}_{= |A_S|}$$

Euler's Totient function ϕ

$\phi(n) =$ # of numbers between 1 and n that are co-prime with n.

$\phi(12) = 4$: 1 $\cancel{2}$ $\cancel{3}$ $\cancel{4}$ 5 $\cancel{6}$ 7 $\cancel{8}$ $\cancel{9}$ $\cancel{10}$ 11 $\cancel{12}$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of n.

$12 = 2^2 3^1$

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$$

$$\phi(12) = 12\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 4$$

$$A = [n]$$

$$\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$$

$$\phi(n) = \{ x : \underbrace{x \notin A_1}_{\text{p_1 is not a factor of x}} \;\&\; \underbrace{x \notin A_2}_{\text{p_2 is not a factor of x}} \;\&\; x \notin A_3 \;\&\; \cdots \}$$

$$A_1 = \{ y \in [n] : p_1 \text{ divides } y \}$$

$$\phi(n) = \sum_{S \subseteq [k]} (-1)^{|S|} \underbrace{|A_S|}_{\frac{n}{\prod_{i \in S} p_i}} = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

Surjections

Fix n, m.

$\# f : [n] \longrightarrow [m]$

s.t. f is <u>onto</u>

$A = \{ f : [n] \longrightarrow [m] \}$

R_f = range of $= \{ y \in [m] : \exists x \text{ s.t. } f(x) = y \}$

$SUR = \{ \text{surjections} \}$

$$SUR = \{ f : \underbrace{1 \in R_f}_{f \notin A_1} \;\&\; \underbrace{2 \in R_f}_{f \notin A_2} \;\&\; \cdots \;\&\; m \in R_f \}$$

$$1 \in R_f \longleftrightarrow f \notin A_1$$

$$1 \notin R_f \longleftrightarrow f \in A_1$$

$$A_1 := \{ f : 1 \notin R_f \} \qquad A_i := \{ f : i \notin R_f \}$$

$$|SUR| = \left| \bigcap_{i=1}^{m} \overline{A_i} \right|$$

$$|S \cup R| = \sum_{S \subseteq [m]} (-1)^{|S|} |A_S|$$

$$\{f : S \cap R_f = \emptyset\}$$

$$= \{f : [n] \to [m] \setminus S\}$$

$$|A_S| = (m - |S|)^n$$

$$|S \cup R| = \sum_{S \subseteq [m]} (-1)^{|S|} (m - |S|)^n = \sum_{k=0}^{m} \binom{m}{k} (-1)^k (m-k)^n$$