8/31/11

Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

"n choose k"

X is a finite set

$$\binom{X}{k} = \{A : A \subseteq X \text{ and } |A| = k\}$$

Thm

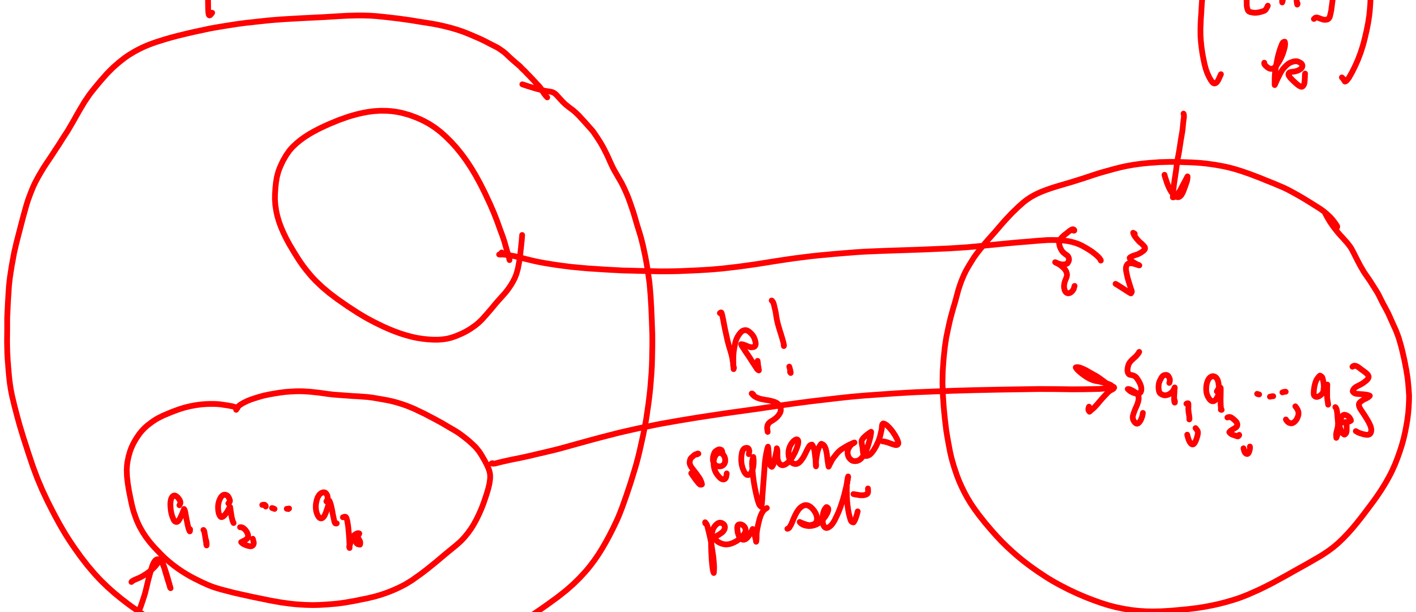
$$\left| \binom{X}{k} \right| = \binom{|X|}{k}$$

n

Proof

$|S| = n(n-1)\cdots(n-k+1)$

$\binom{[n]}{k}$



sequences $a_1 a_2 \cdots a_k$
a_i are distinct & $a_i \in [n]$

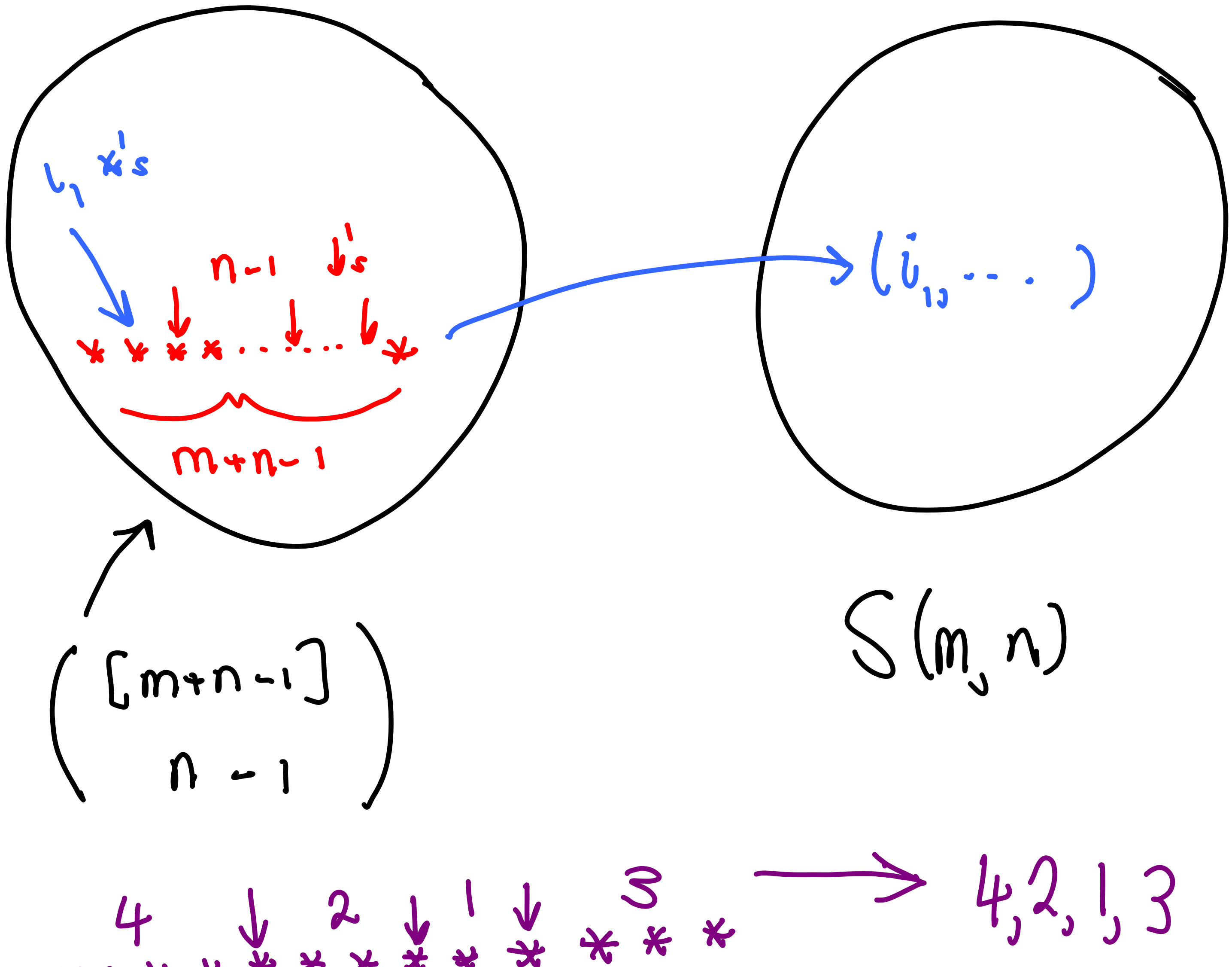
$\binom{[n]}{k} \times k! = |S|$

m, n are positive integers. $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$

$$S(m,n) = \{ (i_1, i_2, \dots, i_n) \in \mathbb{Z}_+^n : i_1 + \dots + i_n = m \}$$

Thm

$$|S(m,n)| = \binom{m+n-1}{n-1}$$



$m=10, \quad n=4$

$\ast \quad \ast \quad \overset{\downarrow}{\ast} \quad \overset{\downarrow}{\ast} \quad \ast \quad \ast \quad \ast \quad \longrightarrow \quad 2, 0, 3$

$m = 5, \quad n = 3$

Going back:

$3, 4, 0, 2$

$\overset{3}{\ast}\ \ast\ \ast\ \overset{\downarrow}{\ast}\ \ast\ \overset{4}{\ast}\ \ast\ \ast\ \overset{\downarrow}{\ast}\ \overset{0}{}\ \overset{\downarrow}{\ast}\ \overset{2}{\ast}\ \ast$

$m = 9, \quad n = 4$

m balls, n colors for balls

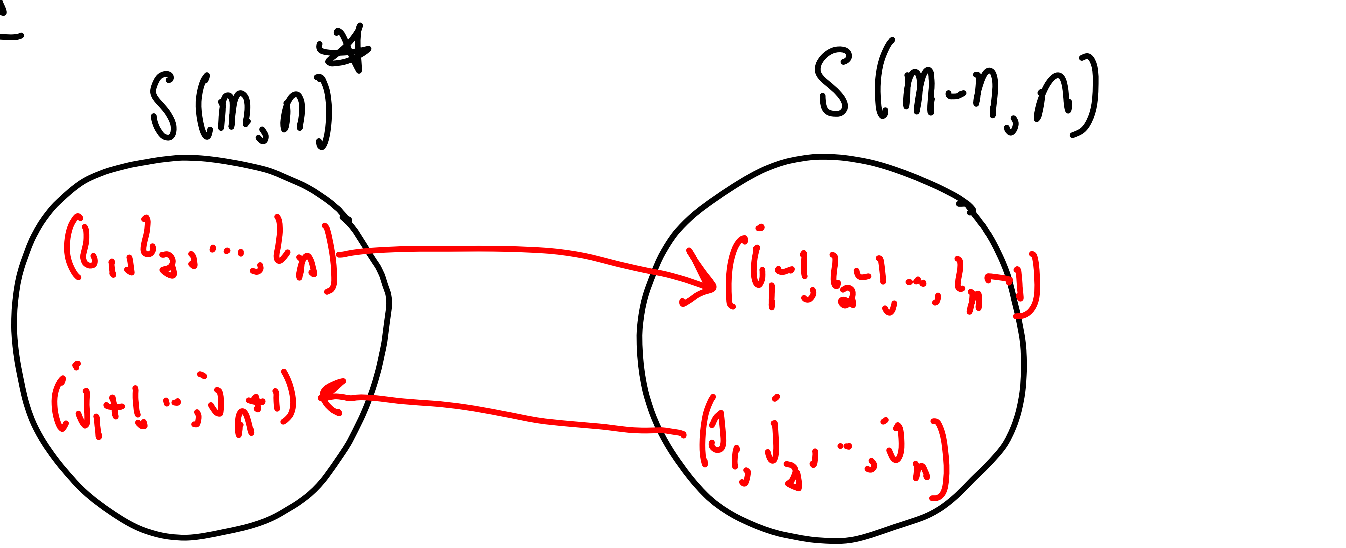
indistinguishable

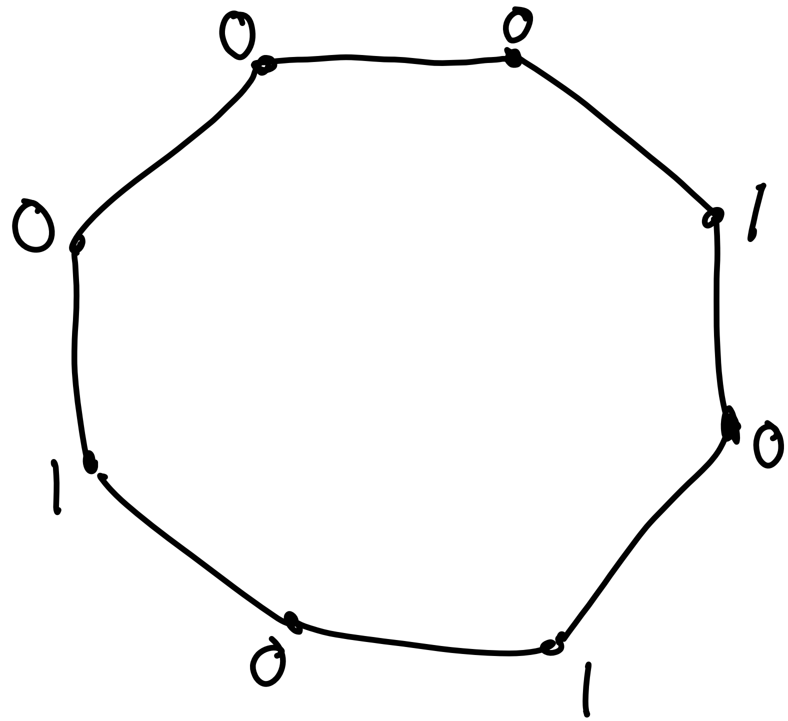
$$S(m,n)^* = \{ (i_1, i_2, \dots, i_n) \in S(m,n) : i_1 \geq 1,\ i_2 \geq 1,\ \dots,\ i_n \geq 1 \}$$

__Thm__

$$|S(m,n)^*| = \binom{m-1}{n-1} = |S(m-n,n)|$$

__Proof__





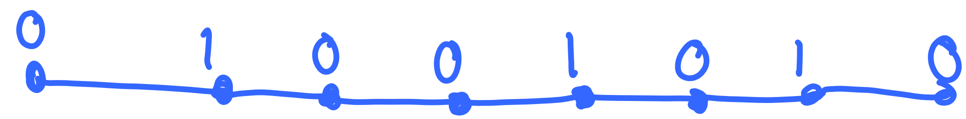
Polygon with n vertices

Place k 1's.

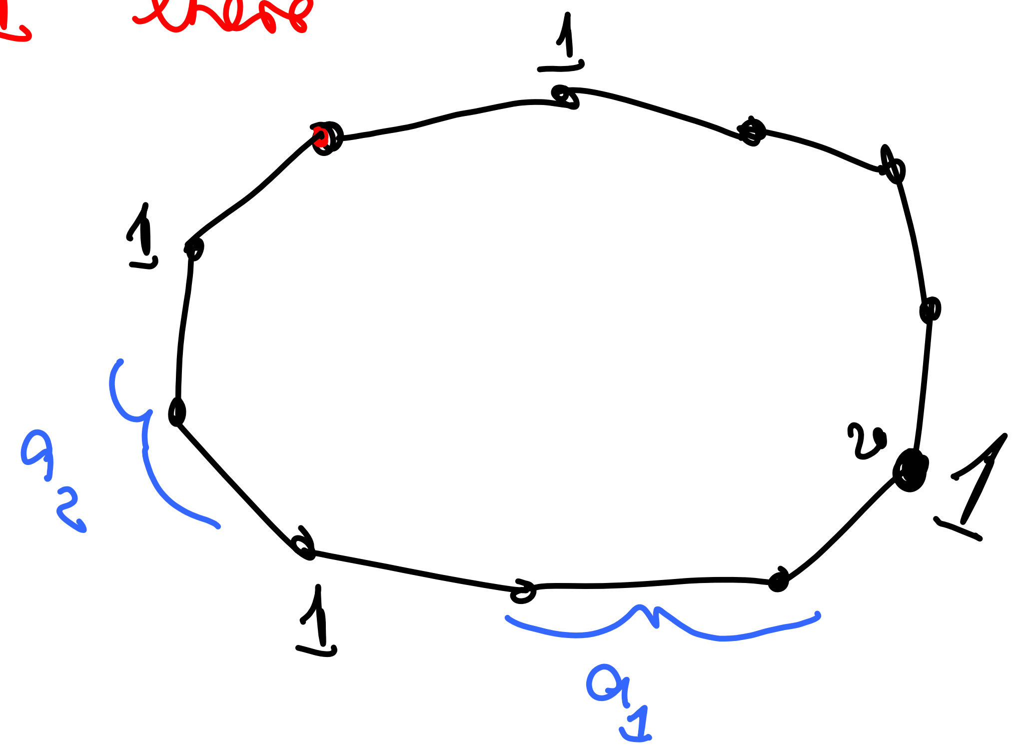
No two 1's are adjacent.

How many ways?

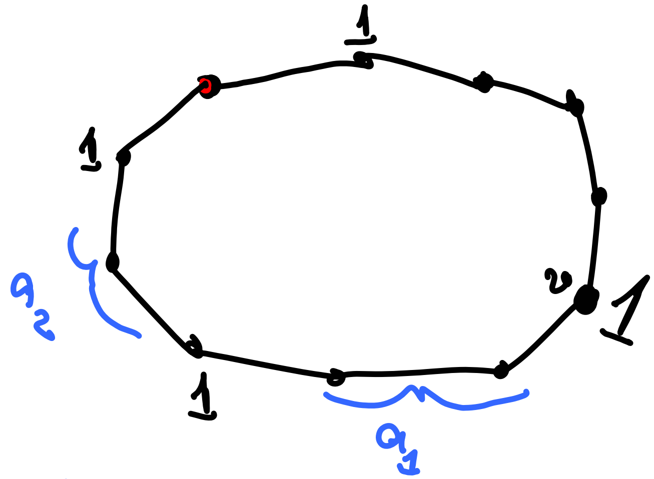
$S(m,n)^*$:



Choose some v on polygon and put a 1 there



Now I choose gaps between next 2 1's.



Now I choose gaps between next 1's.

Match $a_1, a_2, \dots, a_k$ s.t.

(a) $a_i \geq 1 \quad i = 1, 2, \dots, k$

(b) $a_1 + \dots + a_k = n - k$

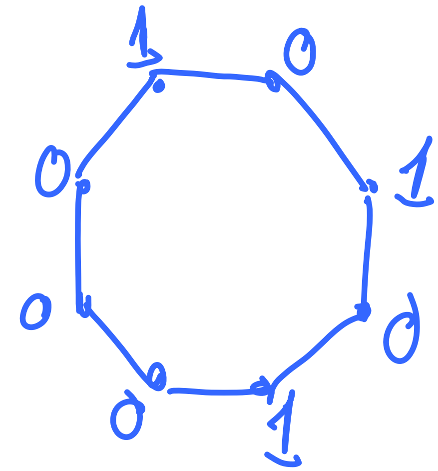
choices for $a_1, \dots, a_k$ is $\binom{n-k-1}{k-1}$

Choose v

Choose $a_1, a_2, \ldots, a_k$

$$\Downarrow$$

$$\# \text{ choices} = n \binom{n-k-1}{k-1}$$



3 choices from left give this

In general, each pattern occurs k times

$$\text{Total} = \frac{n}{k} \binom{n-k-1}{k-1}.$$