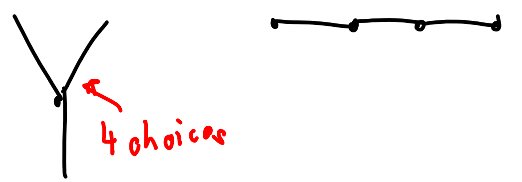
12\2\11

Counting trees – Cayley's Formula

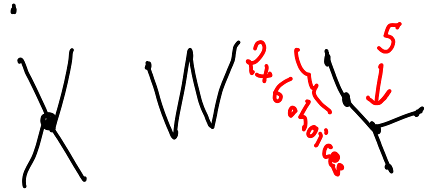
How many distinct spanning trees are there in K_n

$n=4$:



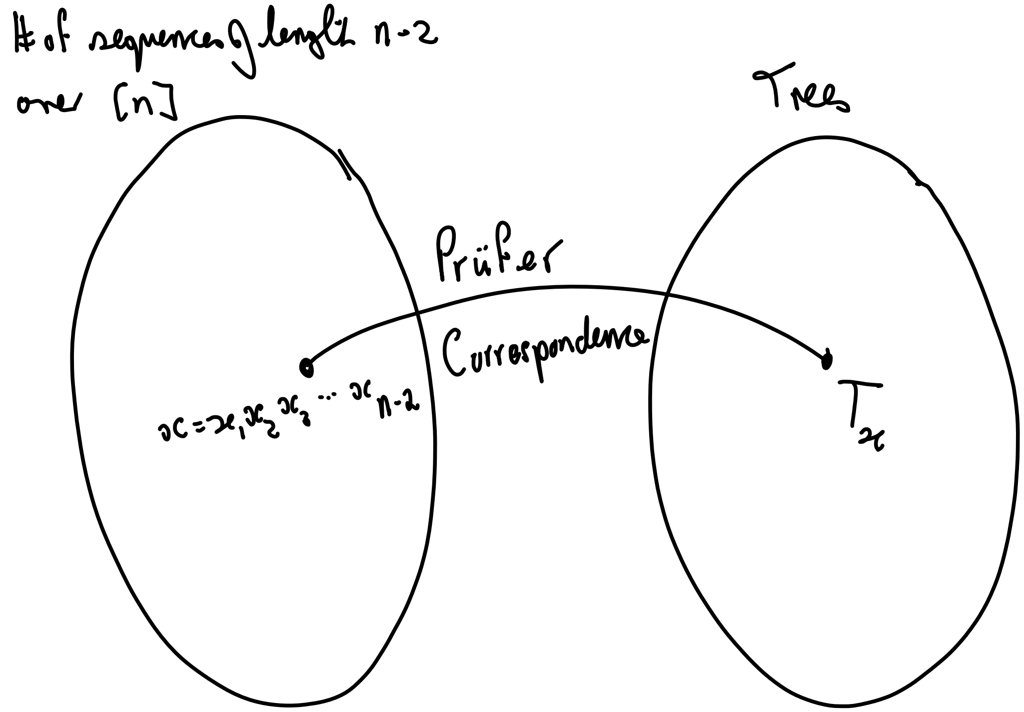
4 $\qquad$ $12 = \dfrac{4!}{2}$

$n=5$:

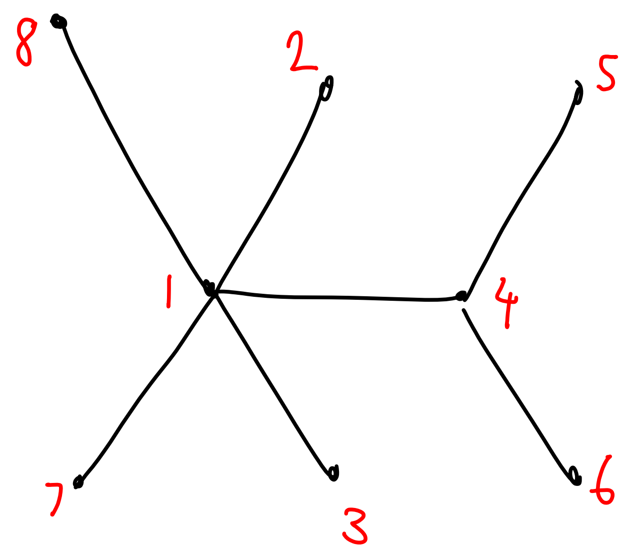


5 $\qquad$ $60 = \dfrac{5!}{2}$ $\qquad$ 60

Thm: there are n^{n-2} spanning trees



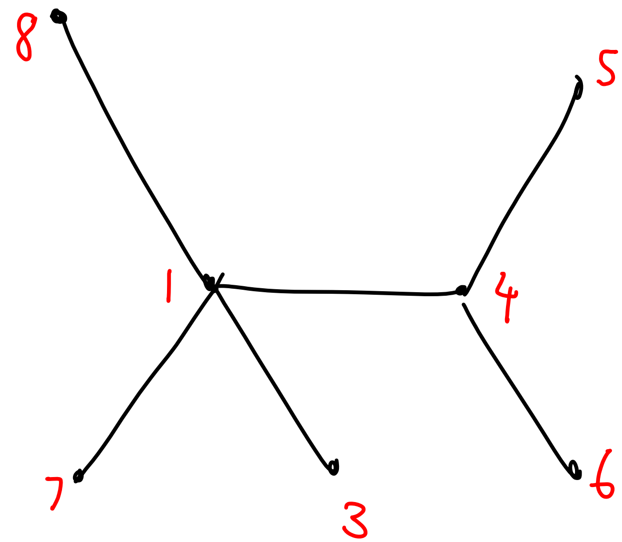
$n=8$



Find smallest numbered leaf, write down its nbr.
Delete leaf:

Leaf	2
Nbr	1
	x_1

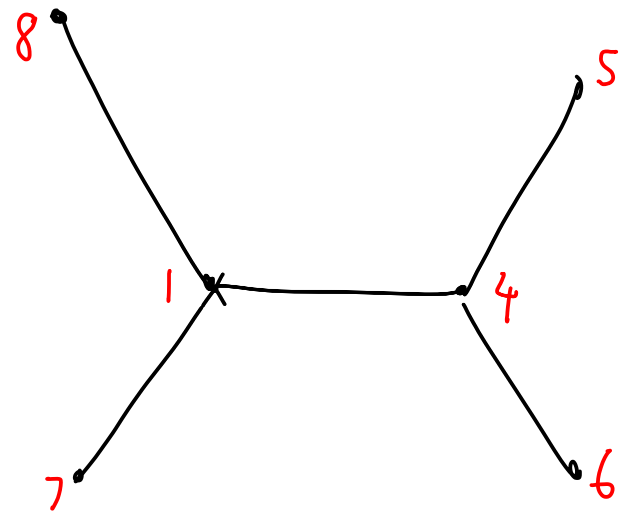
$n=8$



Find smallest numbered leaf, write down its nbr.
Delete leaf:

Leaf	2	3
Nbr	1	1
	x_1	x_2

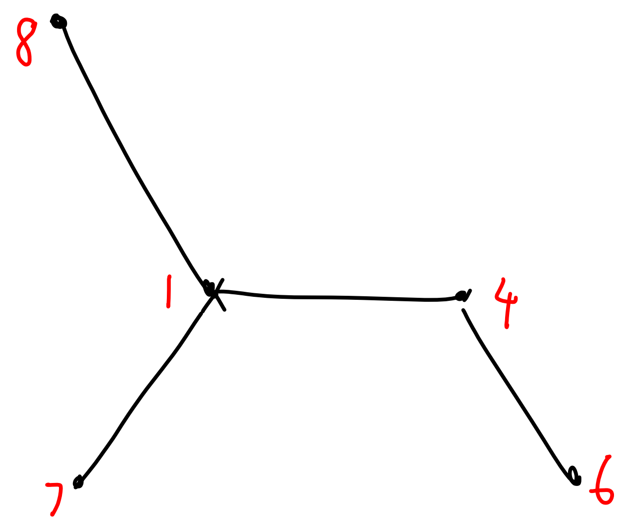
$n=8$



Find smallest numbered leaf, write down its nbr.
Delete leaf:

Leaf	2	3	5
Nbr	1	1	4
	x_1	x_2	x_3

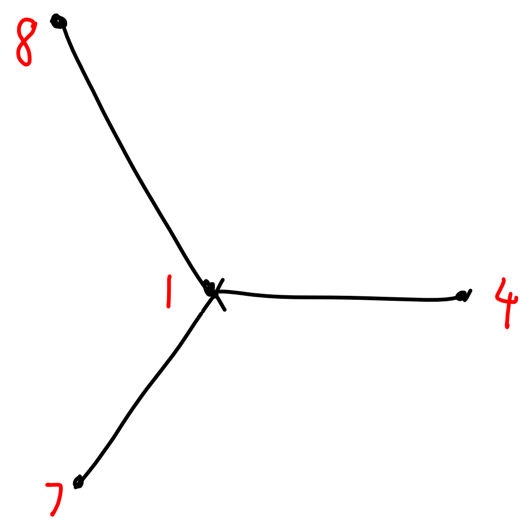
$n=8$



Find smallest numbered leaf, write down its nbr.
Delete leaf:

Leaf	2	3	5	6
Nbr	1	1	4	4
	x_1	x_2	x_3	x_4

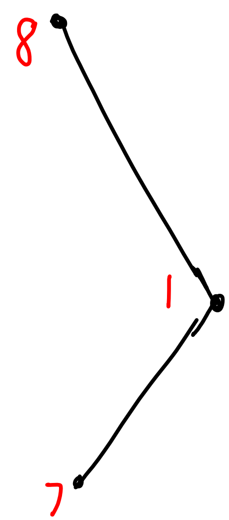
$n=8$



Find smallest numbered leaf, write down its nbr.
Delete leaf:

Leaf	2	3	5	6	4
Nbr	1	1	4	4	1
	x_1	x_2	x_3	x_4	x_5

$n=8$

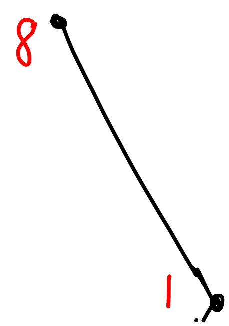


Find smallest numbered leaf, write down its nbr.
Delete leaf:

Leaf	2	3	5	6	4	7
Nbr	1	1	4	4	1	1
	x_1	x_2	x_3	x_4	x_5	x_6

FINISHED

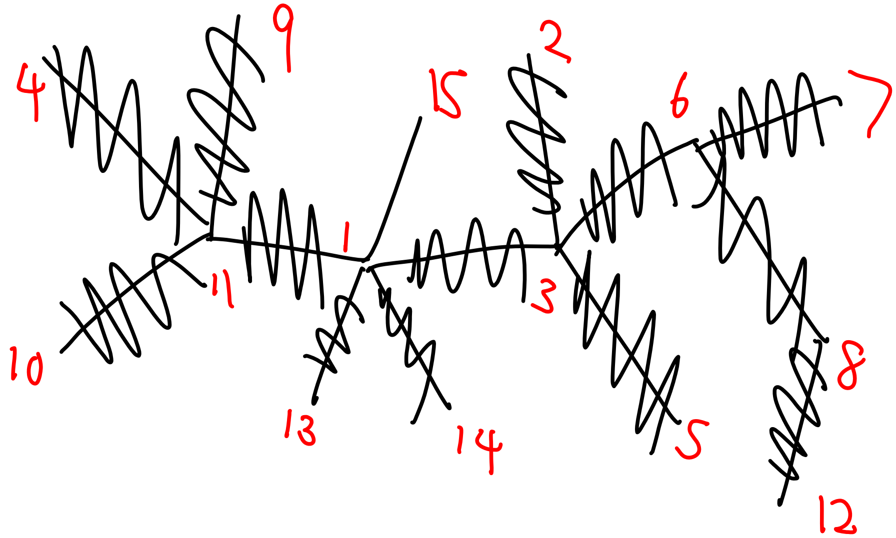
$n=8$



Find smallest numbered leaf, write down its nbr.
Delete leaf:

Leaf	2	3	5	6	4	7
Nbr	1	1	4	4	1	1
	x_1	x_2	x_3	x_4	x_5	x_6

FINISHED



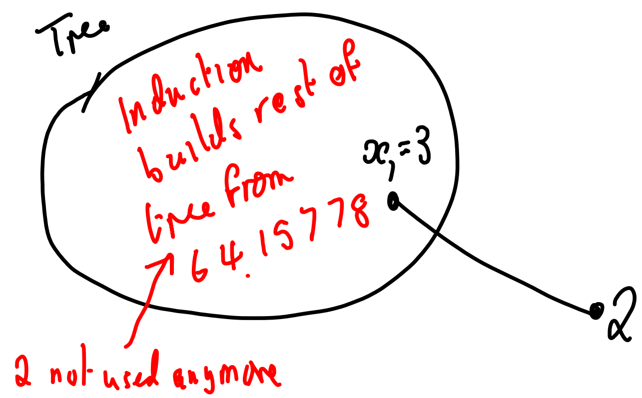
3 11 3 6 11 11 1 8 6 3 1 1 1

$$T \longrightarrow X \quad \text{done}$$

$$X = x_1 \, x_2 \, x_3 \cdots x_{n-2}$$

$n = 10$

3 6 4 1 5 7 7 8

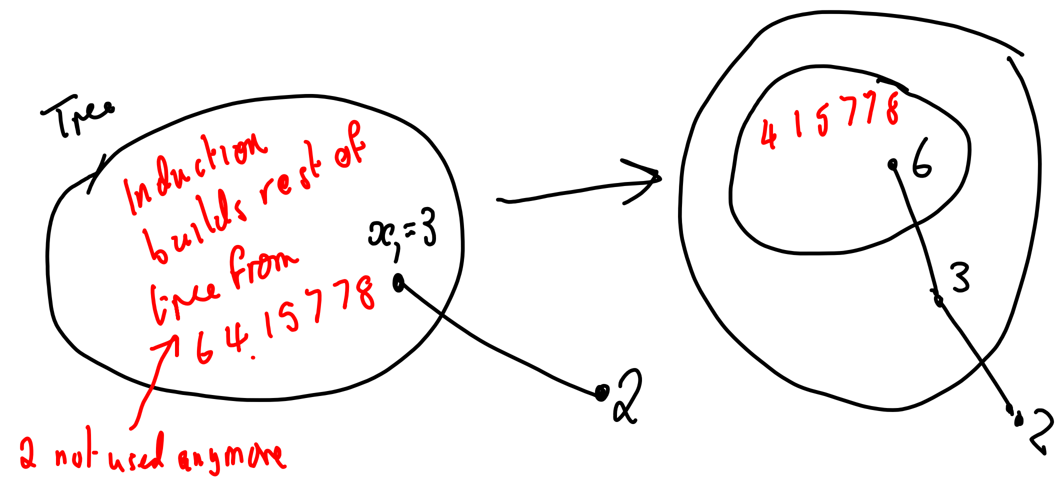


$$T \longrightarrow X \qquad \text{done}$$

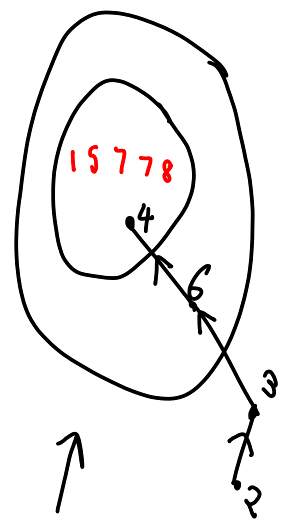
$$X = x_1\, x_2\, x_3 \cdots x_{n-2}$$

$n = 10$

3 6 4 1 5 7 7 8



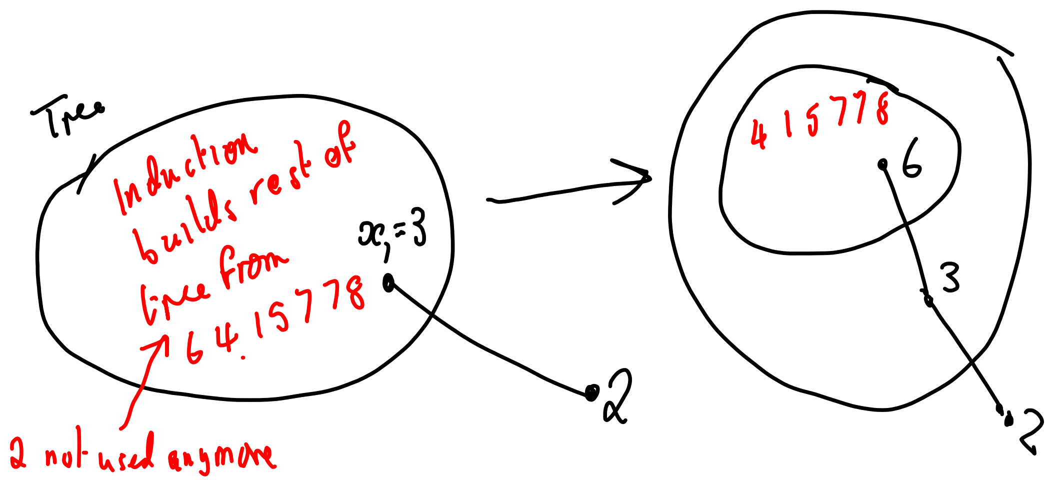
$T \longrightarrow X$ done

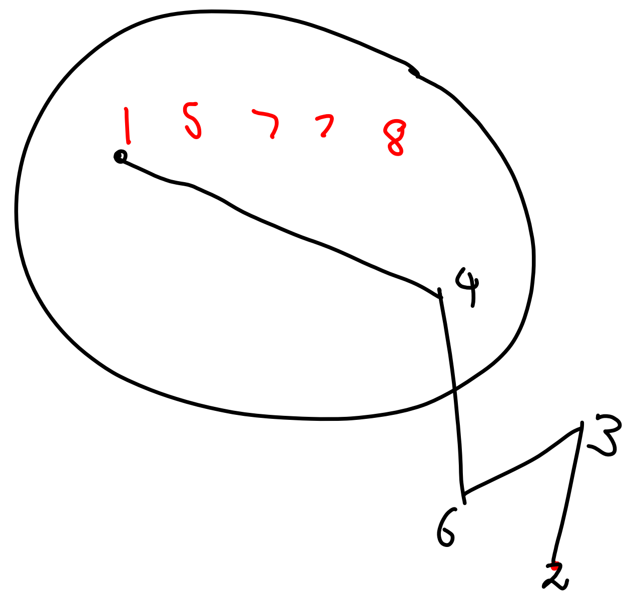


$X = x_1 \, x_2 \, x_3 \cdots x_{n-2}$

$n = 10$

3 6 4 1 5 7 7 8





2,3,6 deleted

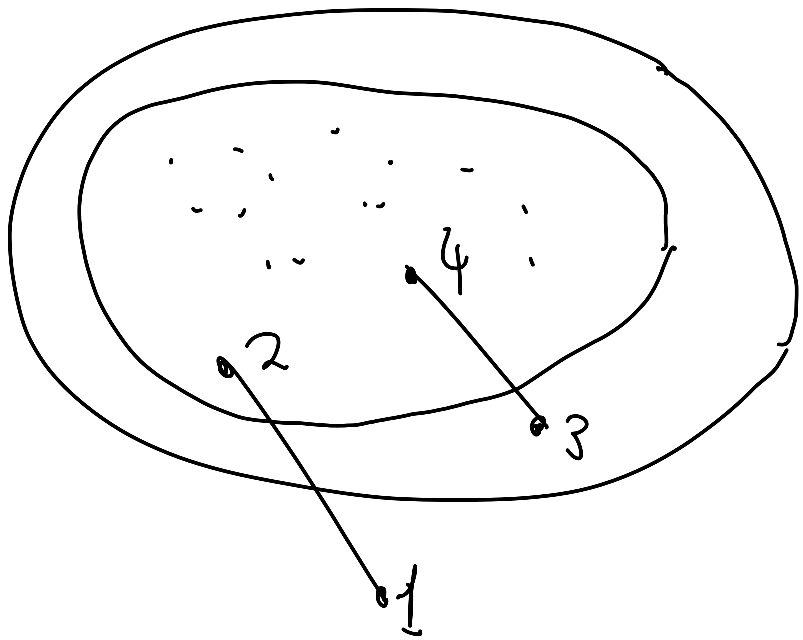
Tree left is on

1,4,5,7,8,9,10

and so on

$n = 10$

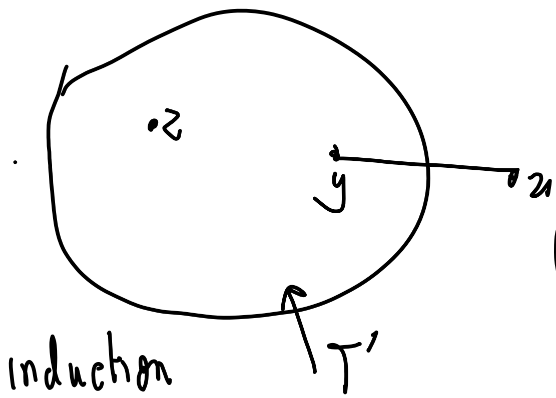
2 4 5 6 2 8 9 10



Observation

If vertex v has degree $d_v = d_v(T)$ then v appears $d_v - 1$ times in the sequence.

(i) True if v has degree one: you only see its Nbr.



(i) z appears $d_z(T') - 1$
$= d_z(T) - 1$ times

(ii) y appears $1 + d_y(T') - 1$ times
$= d_y(T) - 1$

Use induction

The number of trees with degree sequence

$d_1, d_2, \ldots, d_n$ $\quad [d_1+d_2+\cdots+d_n = 2(n-1)]$

$$= \binom{n-2}{d_1-1,\ d_2-1, \ldots, d_n-1}$$

trees with $d_1, \ldots, d_n$

= # sequences over

= # permutations of multi-set

$\{(d_1-1)\times 1,\ (d_2-1)\times 2, \ \ldots \}$