11\9\11

Partially Ordered Sets

A <u>Poset</u> is a set P plus an order relation $\leq$

(i) $a \leq a$ (Reflexive)

(ii) $a \leq b$ & $b \leq c \Rightarrow a \leq c$ (transitive)

(iii) $a \leq b$ & $b \leq a \Rightarrow a = b$ (anti-symmetry)

Examples

(I) $P = \{1, 2, 3, \ldots\}$ and $a \leq b$ has usual meaning

(total order)

(II) $P = \{1, 2, 3, \ldots\}$ and $a \leq b$ iff a divides b

(III) $P = \{\underbrace{A_1, A_2, \ldots, A_m}_{m \text{ elements}}\}$ $\quad A_i \leq A_j$ iff $A_i \subseteq A_j$

$A_i \subseteq A$

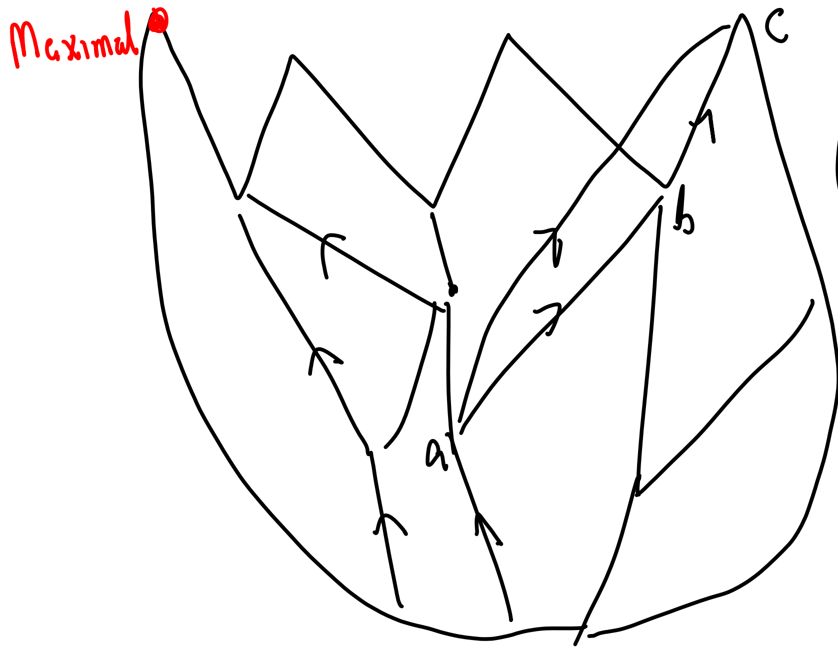
$i = 1, 2, \ldots, m$

(II) $2 \nleq 3$ & $3 \nleq 2$

2 and 3 are incomparable

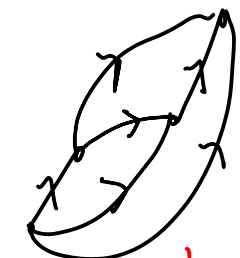
A pair of elements a, b are <u>comparable</u> if

$a \leq b$ or $b \leq a$.



Draw a digraph

$(P, \{(a,b) : a \leq b\})$



A poset without incomparable pairs is called a <u>total</u> order.

$a < b$ if $a \leq b$ and $a \neq b$

A chain is a sequence

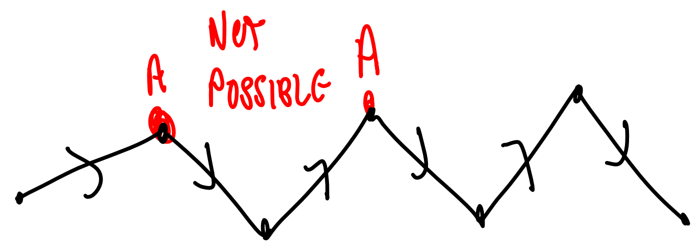
$$a_1 < a_2 < a_3 < \cdots < a_k$$

A set A is an anti-chain if every $a \neq b \in A$ are incomparable

A Sperner family is an anti-chain for example (III).

If C is a chain and A is an anti-chain then

$$|A \cap C| \leq 1$$



C

Questions

cover: every element in one of the sets

(i) Size of largest chain v # of anti-chains needed to cover P

anti-chains needed $\geq$ size of largest chain

(ii) Size of largest anti-chain v # of chains needed to cover P

chains needed $\geq$ size of largest anti-chain

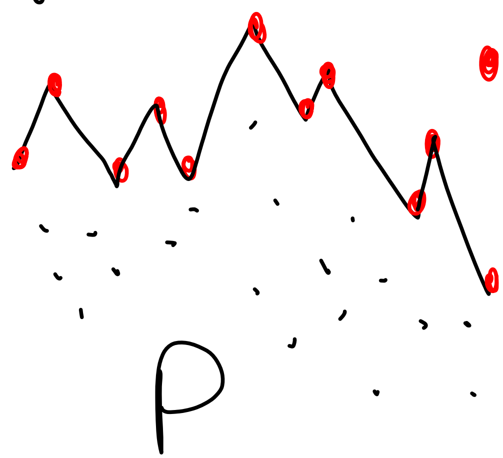
Theorem

min. # anti-chains needed to cover P

$=$ max. size of a chain.

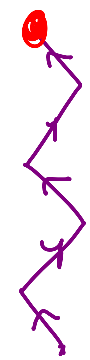
Proof

By induction on length of longest chain $= \ell$



● ≡ maximal element

Every maximal chain ends here



$A_1 = \{\bullet\}$

$P_1 = P \setminus A_1$

longest chain goes down by 1

P_1 can be covered by $\ell - 1$ anti-chains - ind.

Dilworths Theorem

min # chains needed to cover P

= max size of largest anti-chain

Proof - later

Application 1

Erdös - Szekerés Theorem

$a_1, a_2, \ldots, a_n$ are reals.

l^+ = length of longest monotone increasing sequence

l^- = length of longest monotone decreasing sequence

$$l^+ \times l^- \geq n.$$

$$\max\{l^+, l^-\} \geq \left\lceil \sqrt{n} \right\rceil$$

$n = k^2+1$

$$\geq \left\lceil \sqrt{k^2+1} \right\rceil = k+1$$

What is the poset?

$$P = \{ (a_i, i) : i = 1, 2, \cdots, n \}$$

$(a_i, i) \leqslant (a_j, j)$ if $a_i \leqslant a_j$ & $i < j$

Chains give rise to monotone increasing sequences.

Anti-chains give rise to monotone decreasing sequences

$$A = \{ (a_{i_1}, i_1), (a_{i_2}, i_2), \cdots \}$$

$i_1 < i_2 < i_3 < \cdots$ $a_{i_1} > a_{i_2}$ else not anti-chain

Dilworth's Theorem $\Longrightarrow$

$$\underbrace{|\text{longest chain}|}_{c} \times \underbrace{|\text{largest anti-chain}|}_{a} \geq |P|$$

$$P = C_1 \cup C_2 \cup \cdots \cup C_a =$$

$|C_1| \leq c$