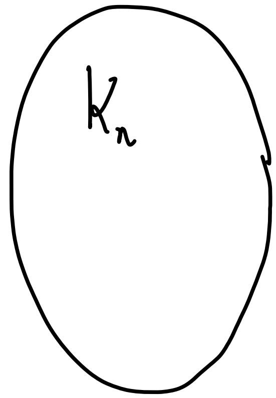
11\4\11

$R(k,1) = R(1,k) = \underline{1}$

$K_1 \leftarrow$ every edge of K_1 is Red

" " " " Blue

$R(2,k) = R(k,2) = k$



$n < k$: color every edge blue

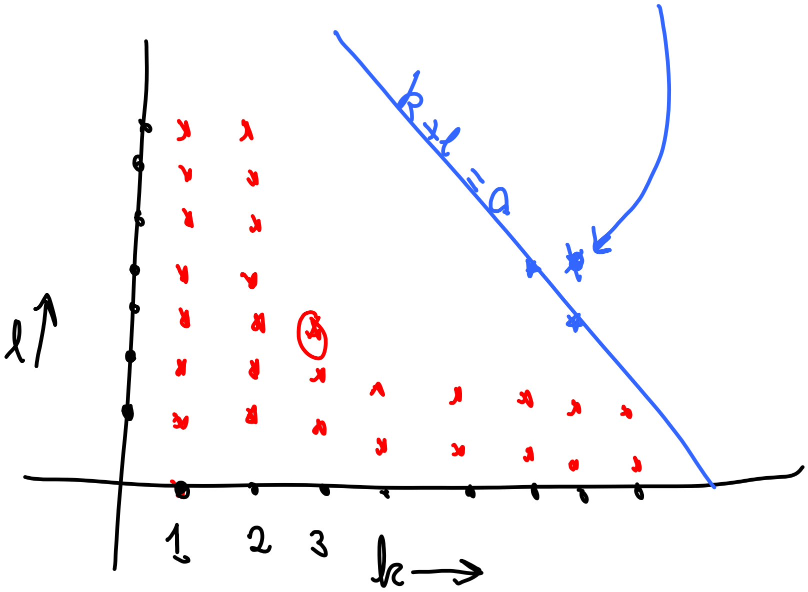
$\Rightarrow R(2,k) > k-1$

$n \geq k$ (i) Only use blue ✓

(ii) Use red → red K_2 ✓

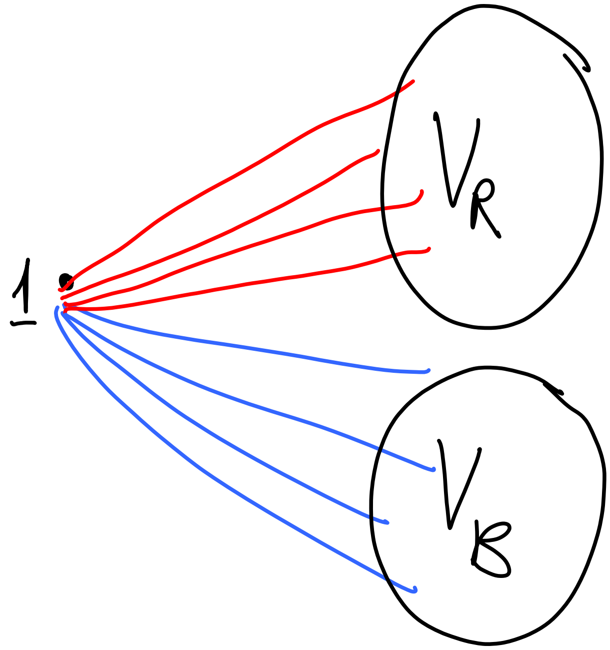
Theorem $R(k,l) \leq R(k,l-1) + R(k-1,l)$

We know
$\times \equiv R(k,l)$ exists



Existence proof by induction on $k+l$

$n \geq R(k,l-1)+R(k-1,l)$



$|V_R|+|V_B| = n-1$

$\Rightarrow |V_R| \geq R(k-1,l)$ (a)

or

$|V_B| \geq R(k,l-1)$ (b)

(a) $\exists$ Red K_{k-1} or $\underbrace{\text{Blue } K_l}_{\checkmark}$ inside V_R

$$R(k,l) \leq \binom{k+l-2}{k-1} \qquad R(k,k) < 2^{2k} = 4^k$$

Current knowledge $R(k,k) \leq \dfrac{4^k}{\phi(k)}$

$\phi(k) = e^{o(k)}$ $\to \infty$ with k

Proof Induction on $k+l$

$$R(k,l) \leq R(k,l-1) + R(k-1,l) \quad \text{just proved}$$

$$\leq \binom{k+l-3}{k-1} + \binom{k+l-3}{k-2} \quad \text{induction}$$

$$= \binom{k+l-2}{k-1} \quad \text{Pascal } \triangle$$

$$R(k,k) > 2^{k/2} = (\sqrt{2})^k$$

$$\sqrt{2} + o(1) \leq \frac{\log_2 R(k,k)}{k} \leq 4 - o(1)$$

$k \to \infty$

Probabilistic method:

If $n \leq 2^{k/2}$ then $\exists$ a 2-coloring of K_n without a Red K_k or a Blue K_k.

We color randomly and show $\Pr(\exists \overset{\text{mono}}{K_n}) < 1$.

$\mathcal{E}_R = \{ \exists$ a Red k-clique $\}$ Want to show

$\mathcal{E}_B = \{ \exists$ a Blue k-clique $\}$ $\Pr(\mathcal{E}_R \cup \mathcal{E}_B) < 1$

$C_1, C_2, \dots, C_N$, $N = \binom{n}{k}$ be the set of all k-cliques.

$\mathcal{E}_{R,j} = \{ C_j$ is Red $\}$, $\mathcal{E}_{B,j} = \{ C_j$ is Blue $\}$

$$
\begin{aligned}
\Pr(\mathcal{E}_R \cup \mathcal{E}_B) &\le \Pr(\mathcal{E}_R) + \Pr(\mathcal{E}_B) \\
&= \Pr\left(\bigcup_{j=1}^{N} \mathcal{E}_{R,j}\right) + \Pr\left(\bigcup_{j=1}^{N} \mathcal{E}_{B,j}\right) \\
&\le \sum_{j=1}^{N} \Pr(\mathcal{E}_{R,j}) + \sum_{j=1}^{N} \Pr(\mathcal{E}_{B,j}) \\
&= \sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} + \sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= 2\binom{n}{k}\left(\frac{1}{2}\right)^{k(k-1)/2}
\end{aligned}
$$

$$
< 2\,\frac{n^k}{k!}\left(\frac{1}{2}\right)^{k(k-1)/2} \le \frac{2^{1+\frac{k^2}{2}-k(k-1)/2}}{k!} = \frac{2^{1+k/2}}{k!} < 1 \text{ for } k \ge 4
$$