11/28/11

## Theorem

Games $G_1, G_2, \ldots, G_p$

Grundy functions $g_1, g_2, \ldots, g_p$

Then if $X = (x_1, \ldots, x_p)$

$$g(x) = g_1(x_1) \oplus \cdots \oplus g_p(x_p).$$

We can assume $p = 2$ and use induction on $p$.

To show

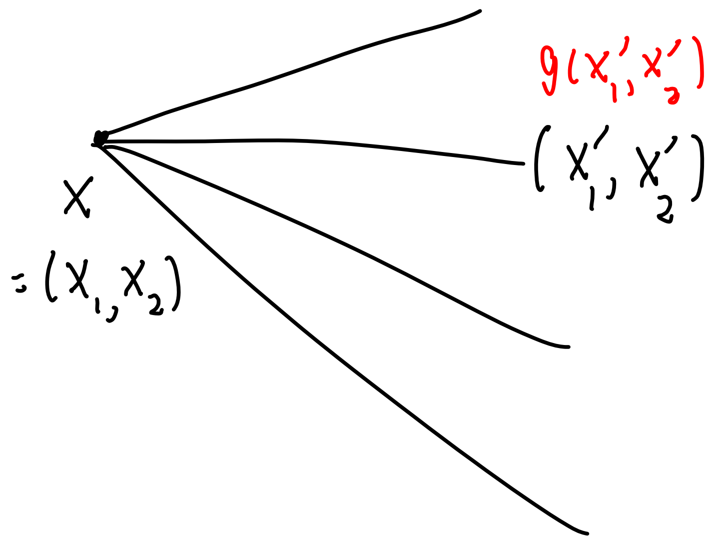
$$g_1(x_1) \oplus g_2(x_2) = \max\{ g_1(x_1') \oplus g_2(x_2'), \; x_1' \in N_1^+(x_1), \, x_2' \in N_2^+(x_2) \}$$

We show

A1: If $x \in X = X_1 \times X_2$ and $g(x) = b > a$ then $\exists \; x' \in N^+(x)$ such that $g(x') = a$

A2: If $x \in X$ and $g(x) = b$ and $x' \in N^+(x)$ then $g(x') \neq b$

A3: If $x \in X$ and $g(x) = 0$ and $x' \in N^+(x)$ then $g(x') \neq 0$.




$$g(x'_1, x'_2) = g_1(x'_1) \oplus g_2(x'_2)$$

A1: $g(x) = b > a$ $\qquad a = (a \oplus b) \oplus b$

Write $\quad a = \underbrace{d}_{a \oplus b} \oplus g_1(x_1) \oplus g_2(x_2)$

We show

(i) $d \oplus g_1(x_1) < g_1(x_1)$ OR (ii) $d \oplus g_2(x_2) < g_2(x_2)$

$\Downarrow$

$\exists\, x_1' \in N^+(x_1)$ s.t.

$g_1(x_1') = d \oplus g_1(x_1)$

$\Rightarrow \quad a = g_1(x_1') \oplus g_2(x_2) = g\overbrace{(x_1', x_2)}^{\in N^+(x)}$

$d = a \oplus b$ and $a < b$

Suppose

binary expansion is back to front

| | | | | | | | | | | $k$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | . | | - | - | | |
| $d$ in binary | x | x | x | x | x | x | x | x | x | 1 | 0 ⋯ 0 ⋯ |
| $a$ | x | x | x | x | x | x | x | x | x | 0 | ⋯⋯ |
| | | | | | | | | | | | same |
| $b$ | x | x | x | x | x | x | x | x | x | 1 | ⋯⋯ |

$d_k = a_k \oplus b_k$ and $a < b = g_1(x_1) \oplus g_2(x_2)$

Assume $g_1(x_1)$ has a 1 in position $k$

$\Rightarrow \quad d \oplus g_1(x_1) < g_1(x_1)$ : "adding" $d$ "kills" this 1

$G_i$ = Game 1, $i = 1, 2, \ldots, 5$

Piles:

| | | | | |
|---|---|---|---|---|
| 34 | 17 | 22 | 136 | 475 |

$$g = 4 \oplus 2 \oplus 2 \oplus 1 \oplus 0$$

$$\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\hline
1 & 0 & 1
\end{array} \neq 0$$

Winning Position

One winning move.
Remove 3 from pile 1.

# More complex subtraction game

(i) First move cannot take all chips

(ii) If previous move took $x$ chips then you can only take $y \leq x$ chips.

Losing positions are powers of 2.

Suppose $n = 52$

$= 1\,1\,0\,1\,0\,0$

Winning strategy: Remove least significant bit

$\Downarrow$

$1\,1\,0\,0\,0\,0$

Opponent cannot reduce the number of 1s in the expansion from what he/she sees.

# Generalisation

We have $f: \mathbb{N} \longrightarrow \mathbb{N}$

(i) $f$ is non-decreasing

(ii) $f(x) \geq x$

If a player takes $x$ then next player takes $y \leq f(x)$.