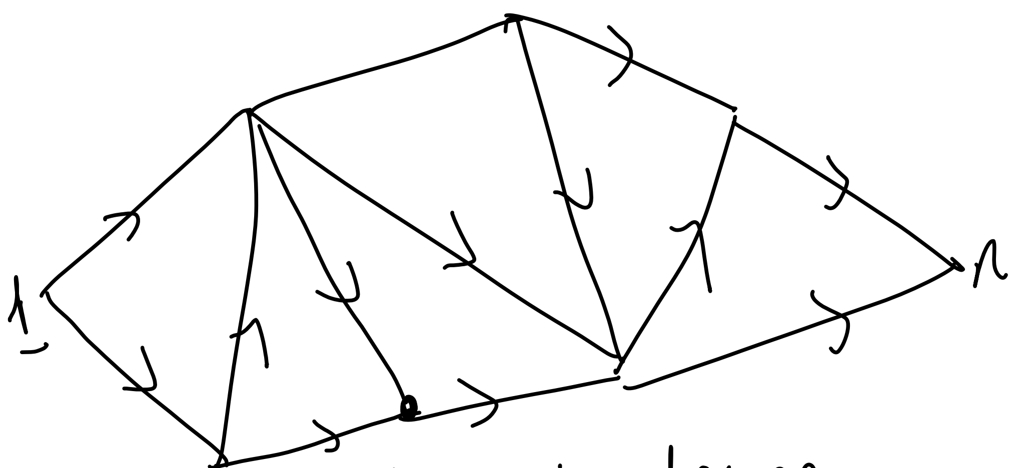
11\21\11

Game $\equiv D = (X, A)$ DAG

Labelling procedure to compute N, P



Label n with P

$n-1$

$n-2$

$\vdots$

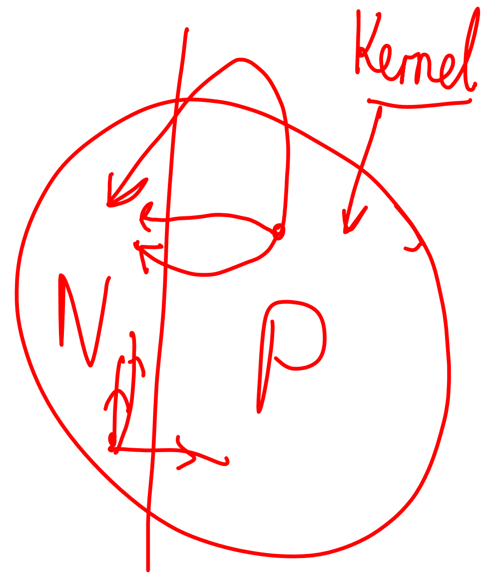
1

Have a topological ordering

all these have been labelled

label

$x \in N$ iff $N^+(x) \cap P \neq \emptyset$



Game 1

Take away game: $S = \{1, 2, 3, 4\}$

$P = \{0, 5, 10, 15, \ldots\}$

$x = 5k + i$, $i = 1, 2, 3, 4$ Remove i $\longrightarrow P$

$x = 5k$, Remove $i \longrightarrow 5(k-1) + (5-i) \in N$

The partition into N, P is unique.

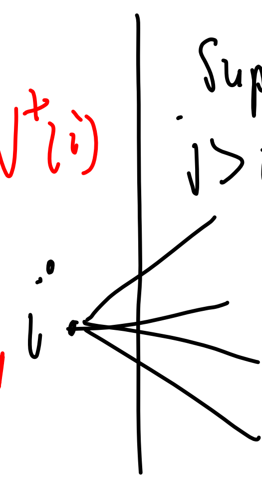
Suppose you had another partition N', P' that satisfied $x \in N'$ iff $N^+(x) \cap P' \neq \emptyset$

Use backward induction on the topological ordering

If $i \in N$ then $\exists j \in N^+(i)$ such that $j \in P$

$\Rightarrow j \in P'$

$\Rightarrow i \in N'$

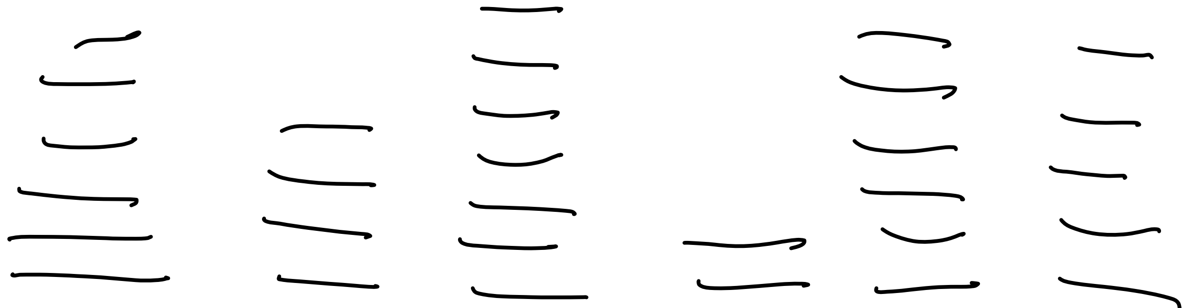


Suppose $j > i \Rightarrow j \in P \cap P'$

or $j \in N \cap N'$

$i \in P \cap P'$

Sums of games (Generalise Nim)



Nim: have piles $X_1, X_2, \ldots, X_n$

To move, select an non-empty pile and remove some chips.

This is the sum of n one pile games.

~~m piles~~ $\quad G_i = (X_i, A_i)$

$G_1 \quad G_2 \quad G_3 \quad \cdots \quad G_m$

Position: $x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_m$

$x = (x_1, x_2, \ldots, x_m) \in X_1 \times X_2 \times \cdots \times X_m$

Move: choose a game G_i, where x_i is not a sink and replace x_i by $x_i' \in N_i^+(x_i)$ i.e. move in G_i.

What do we need to know about $G_1, \ldots, G_m$ to play $G_1 + \cdots + G_m$?

(Sprague)-Grundy Numbering

Grundy number for DAG's

$S \subseteq \{0, 1, 2, \dots \quad \}$

$$\text{mex}(S) = \min\{x \geq 0 : x \notin S\}$$

minimum excluded

$\text{mex}\{1, 5, 12\} = 0$

$\text{mex}\{0, 1, 3, 4, 5\} = 2$

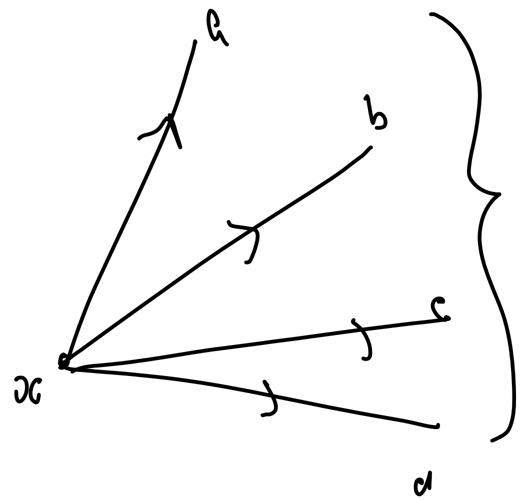
$\vdots$

$x \in P$ iff $g(x) = 0$

because

$g(x) = 0$ iff

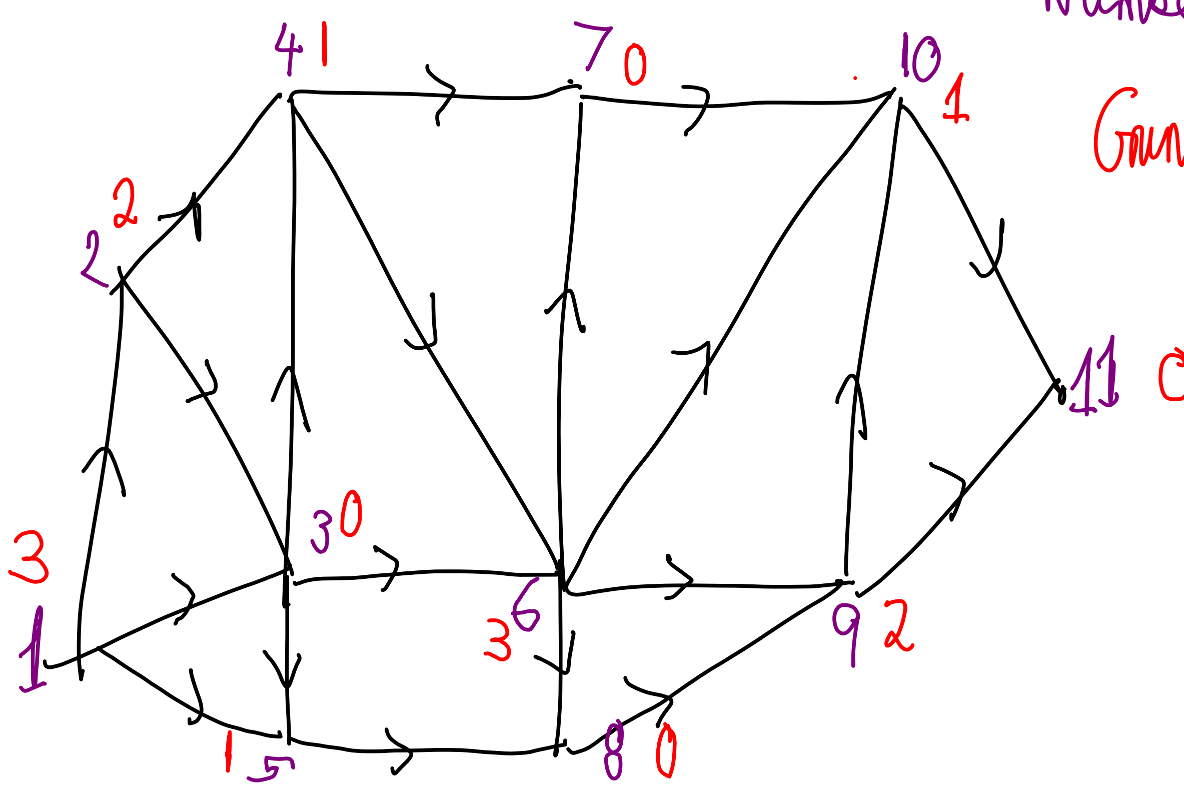
$g(y) > 0, \forall y \in N^{+}(x)$



$$g(x) = \text{mex}\{g(a), g(b), g(c), g(d)\}$$

Topological Number

Grundy Number



For Nim, $g_i(x) = x$

Usefulness:

Sum of game $G_1, G_2, \ldots G_n$

$g_1 \quad g_2 \quad g_n$

$$g(x_1, x_2, \cdots, x_n) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_n(x_n)$$

$a \oplus b$ = bitwise addition without carry

$a = a_1 a_2 \cdots a_k$ $\quad a_i = 0$ or 1

$b = b_1 b_2 \cdots b_k$ $\quad c_i = 0$ iff $a_i = b_i$

$a \oplus b = c_1 c_2 \ldots c_k$

A substraction game

$S = \{2, 4, 5\}$

n	0	1	2	3	4	5	6	7	8	9	10
$g(n)$	0	0	1	1	2	2	3	0	0	1	1

n	11	12	13	14
$g(n)$	2	2	3	0

......

$g(n) = g(n \bmod 7)$