11\18\11

Combinatorial Games

Take-away games

2 players A & B. A goes first.

Players alternately remove some chips.

In this version "some = 1, 2, 3 or 4"

The winner is the player who takes the last chip - i.e. makes the last move.

n	Winner
0	B
1	A
2	A
3	A
4	A
5	B
6	A
$\vdots$	
103	A

← A chooses x, left with $5-x \in 1, 2, 3, 4$ which is a winning position

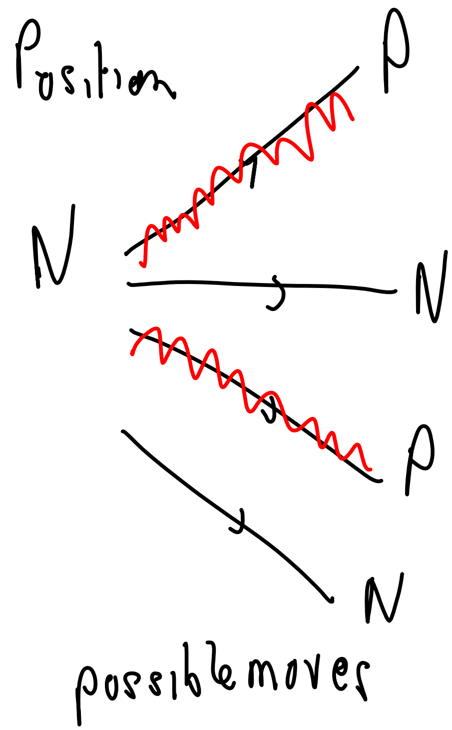
A chooses 1, 5 left which is a losing position

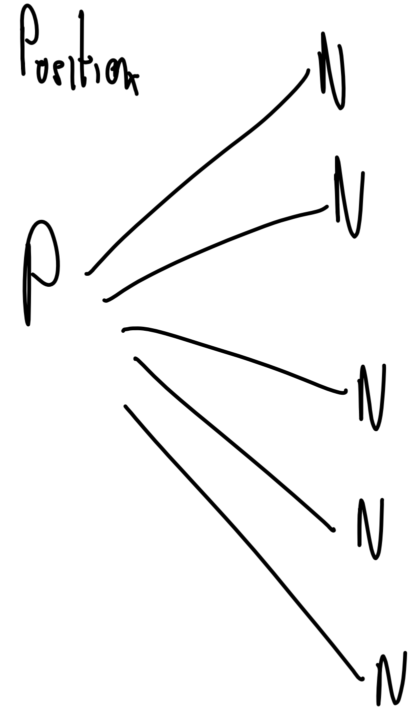
There are two sorts of position:

N position, Next player will win, if he/she plays properly

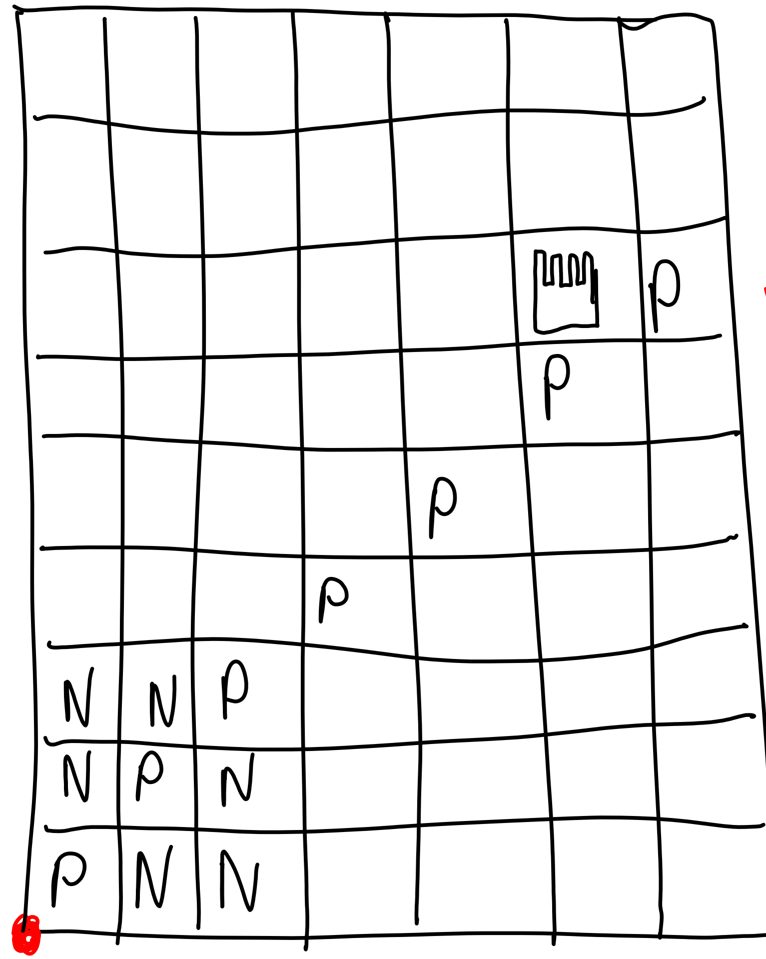
P position Previous player will win if he/she plays properly.

Goal: determine the sets N & P and how to play.





Choose to put next player
into a P position

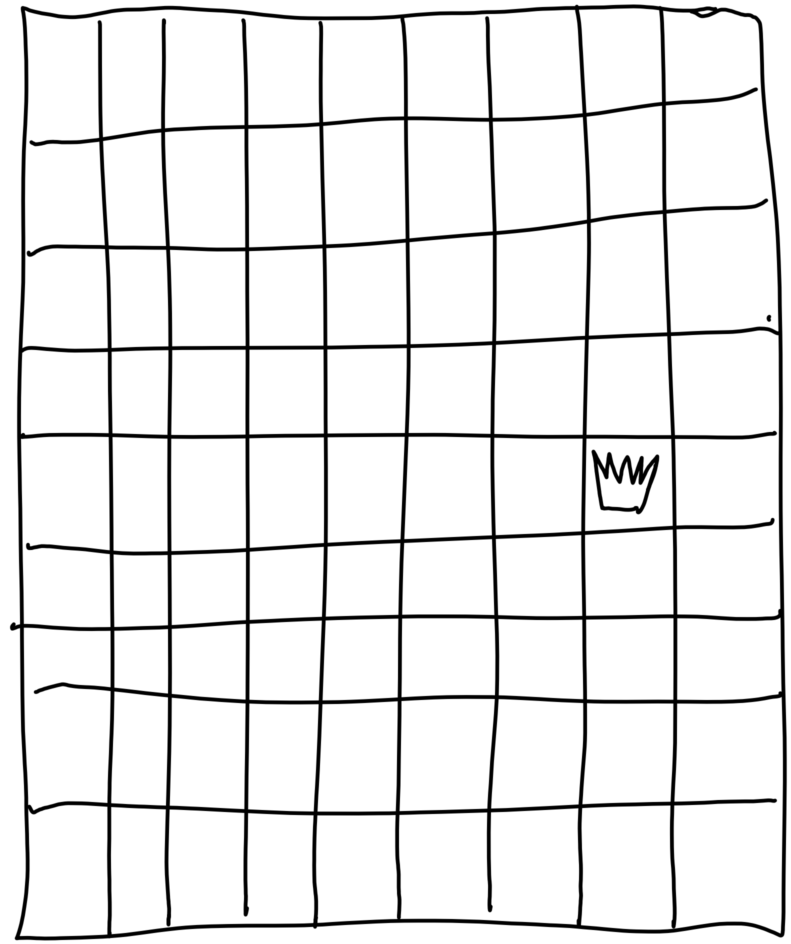


Move piece

↓ or ←

2-pile Nim

General Nim: more dimensions



Move

Wythoff Nim

Geography

$W = \{words\}$

Players alternately choose $w_1, w_2, \dots w_k, \dots \in W$ without replacement so that w_{i+1} starts with last letter of w_i

$W = \{$ countries represented in the U.N. $\}$

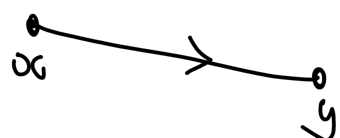
Albania, Argentina, Afghanistan, Netherlands, Sweden, Nepal, Libera, Algeria, Australa, Armenia, Austria, Azerbaijan, Norway, Yemen, Nigeria, Andorra, ...

Reduce to matchings in graphs

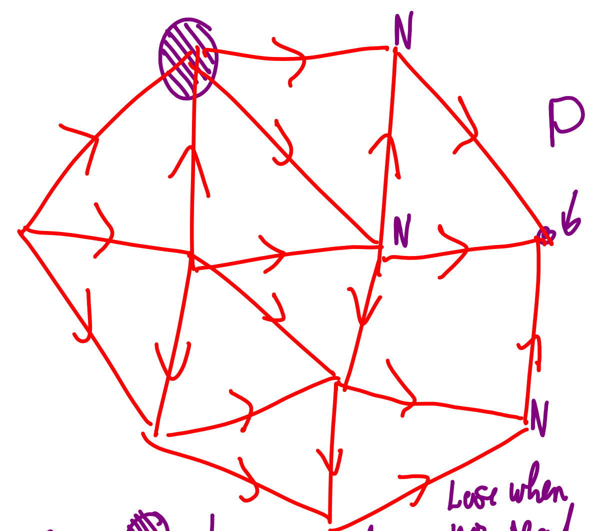
Mathematical Abstraction

Represent each position of the game by a vertex of a digraph $D = (X, A)$

$X = \{\text{positions}\}$



if you can go from x to y in one move

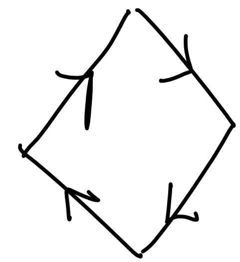


Move: move (circle) along an edge.

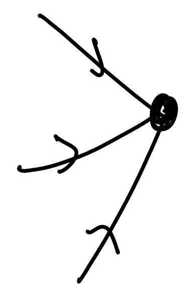
Lose when you reach a "sink"

Assume D is acyclic : DAG

No <u>directed</u> cycles

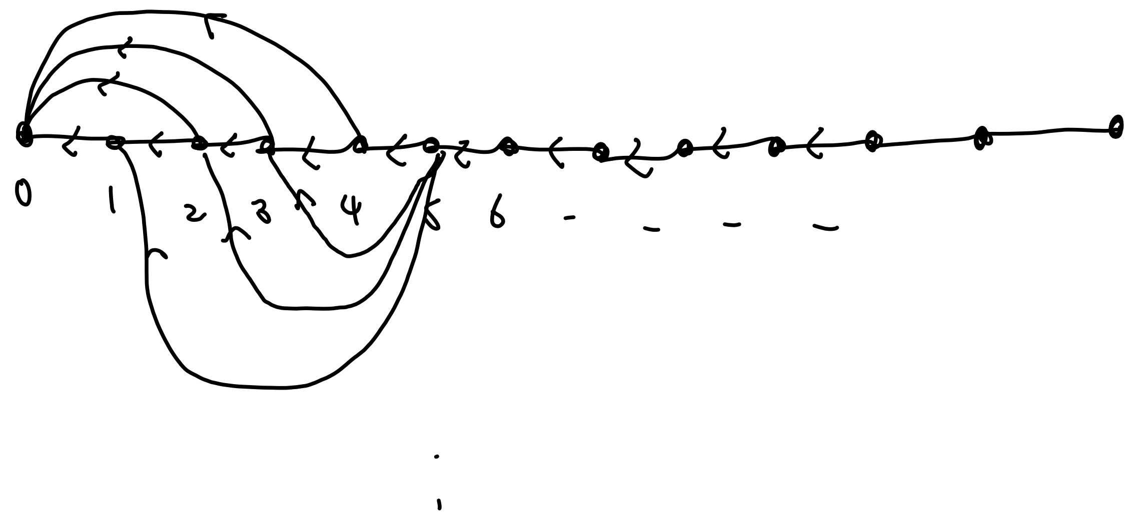


Game ends at a sink



- outdegree 0

Game 1



We want to show next that the game must end.

A <u>topological</u> numbering of D is a

map $\quad f : X \longrightarrow [n] \qquad n := |X|$

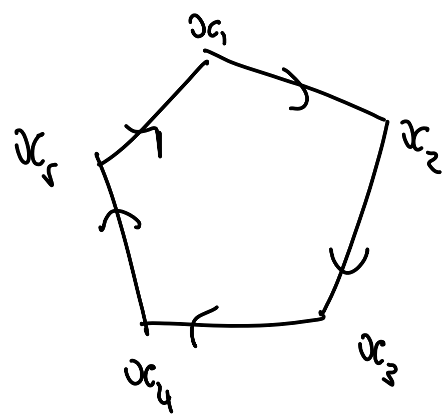
such that

$$x \longrightarrow y \quad \Longrightarrow \quad f(x) < f(y).$$

<u>Thm</u>

D has a topological numbering iff D is a DAG.

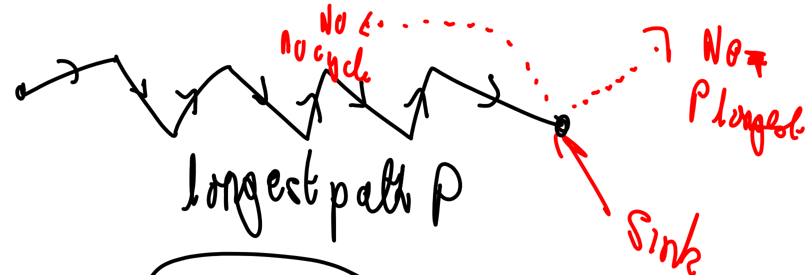
(i)



No ordering

$f(x_1) < f(x_2) < \dots < f(x_5) < f(x_1)$

(ii) DAG: (i) $\exists$ at least one sink



(ii) Numbering