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Dilworth's Theorem:

min # chains to cover = $\underbrace{\text{max size anti-chain}}_{\mu}$

Proof

$C = x_1 < x_2 < \cdots < x_p$ is a maximal chain.

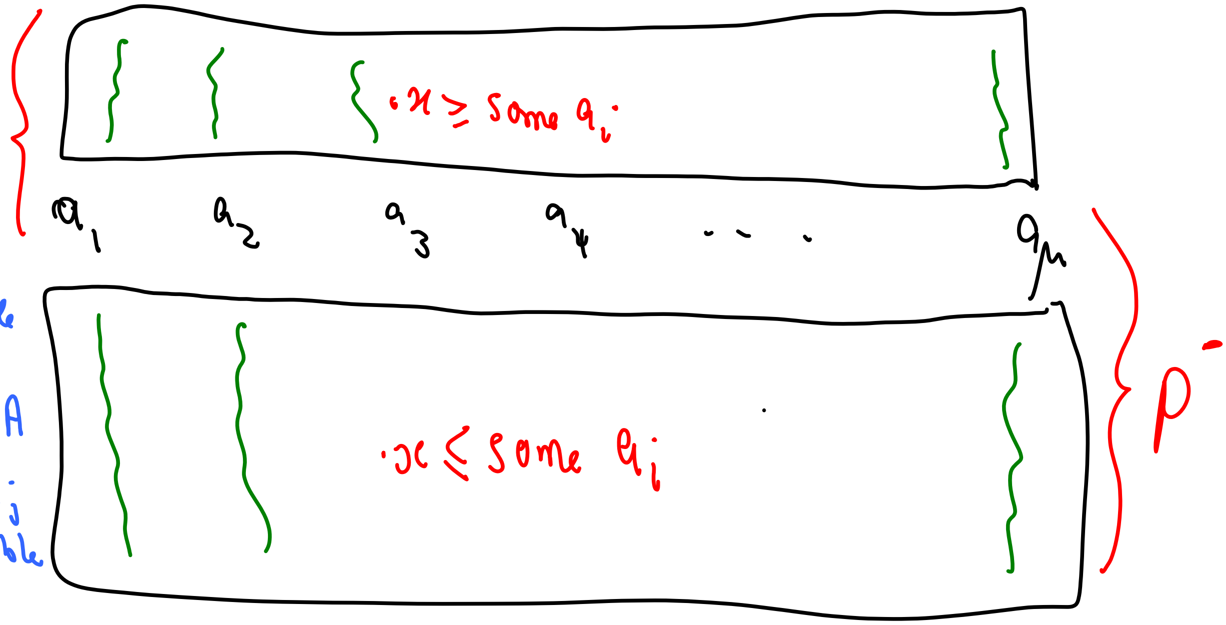
Case 1

Maximum anti-chain size in $P \setminus C$ is $\mu - 1$.

By induction $P \setminus C$ can be covered by chains $C_2, C_3, \ldots, C_\mu$

P can be covered by $C, C_2, \ldots C_\mu$.

Case 2 $P \setminus C$ has an anti-chain $a_1, a_2, \dots, a_m \leftarrow A$



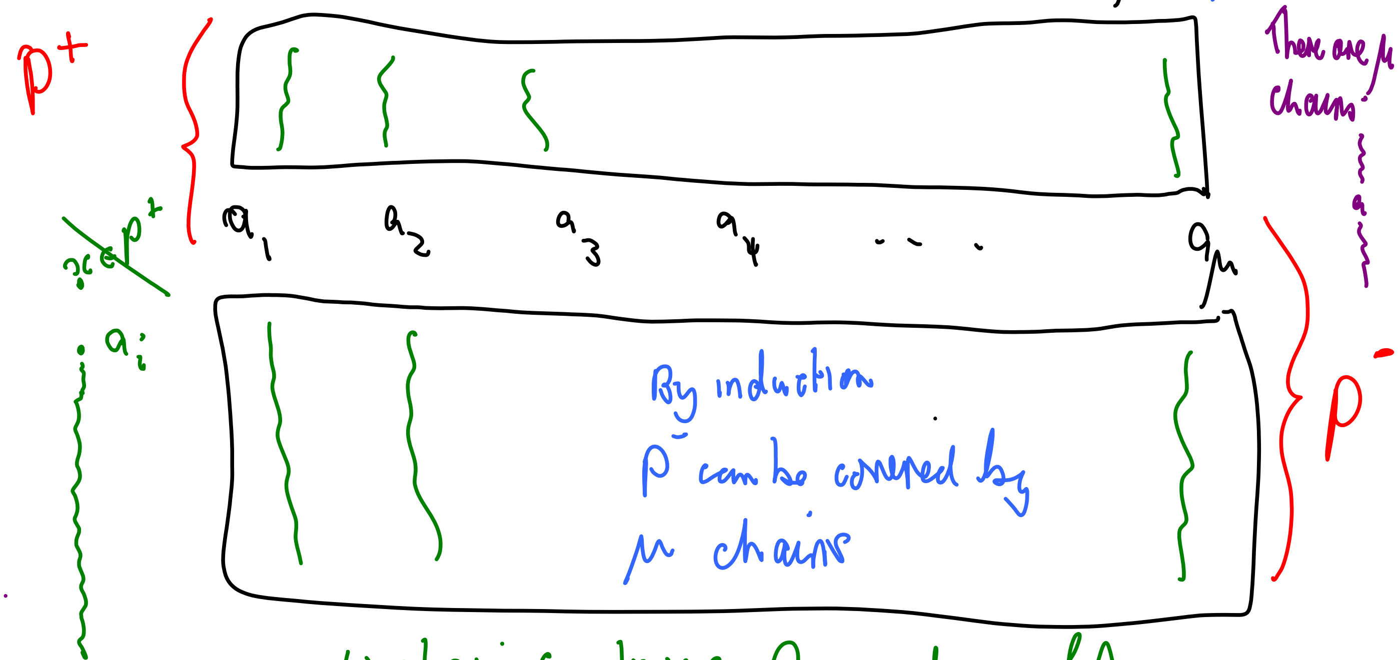
(i) $P = P^+ \cup P^-$
else $\exists x$ incomparable with A

(ii) $P^+ \cap P^- = A$

$a_i < x < a_j$ not possible

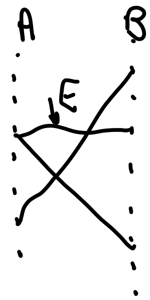
(iii) $x_p \notin P^-$ else you can add some a_i to make C longer

Case 2 $P \setminus C$ has an anti-chain $a_1, a_2, \ldots, a_\mu \leftarrow A$



No chain contains 2 members of A

Hall's Theorem



Assume
$|N(S)| \geq |S| \quad \forall S \subseteq A$

$P = A \cup B$ and $a \leq b$ iff $a \in A, b \in B$ & $(a,b) \in E$

$X = \{a_1, a_2, \dots, a_h, b_1, b_2, \dots, b_k\}$ is largest anti-chain in P

$s = h + k.$

$N(\{a_1, \dots, a_h\}) \subseteq \{B \setminus \{b_1, \dots b_k\}\}$ else X not anti-chain

$\Rightarrow |B| - k \geq h$

$|B| \geq h + k = s$

By Dilworth's Theorem we can cover P with $h+k$ chains:

<u>Chains</u>: matching M of size m

+ $|A|-m$ members of A

+ $|B|-m$ members of B

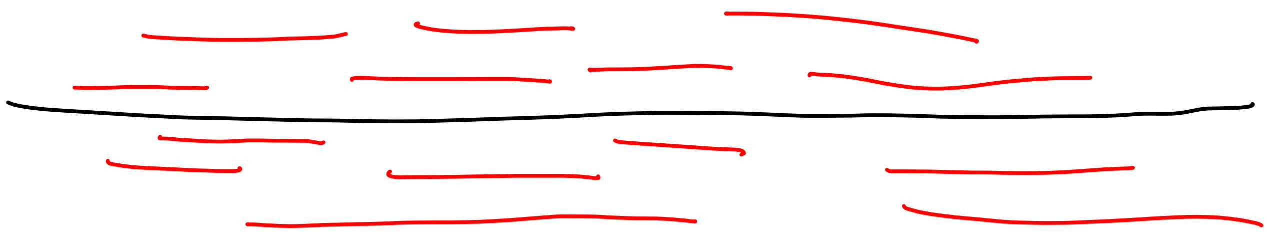
fewest chains = biggest matching

$$\#\text{chains} = m+|A|-m+|B|-m = s \leq |B|$$

$$\Downarrow$$

$$m \geq |A|$$

$I_1, I_2, \ldots\ I_{mn+1}$ are closed intervals on the real line. $\quad I_j : [a_j, b_j]$



Either we have "many" disjoint intervals or many intervals have a common point.

Define a partial ordering

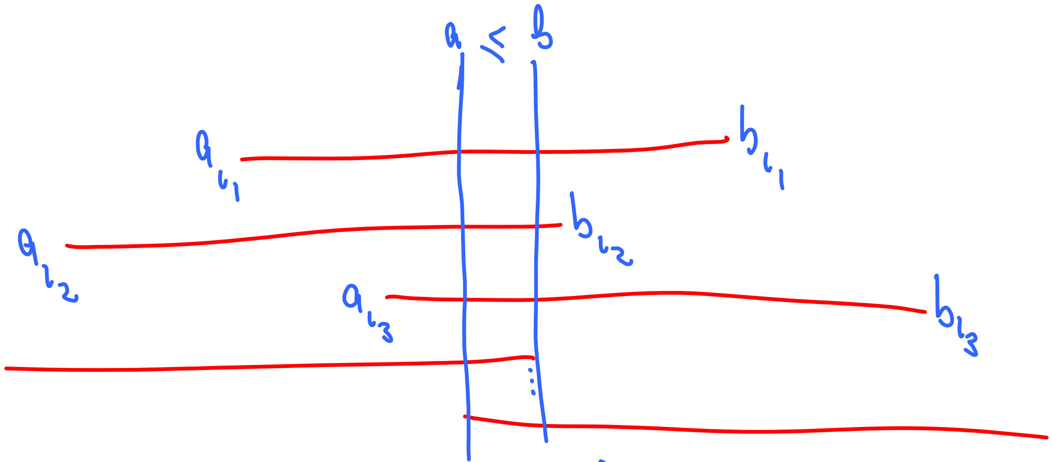
___ < ___

Chain: ___ ___ ___ ___ ___ ___ ___

Suppose maximum chain length is m

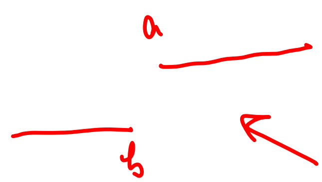
$\Longrightarrow \exists$ (Dilworth) an anti-chain of size $\left\lceil \frac{mn+1}{m} \right\rceil$ need this # of chains to cover poset.

$\exists$ anti-chain of size $n+1$



$a = \max\{a_{i_1}, a_{i_2}, \dots\}$

$b = \min\{b_{i_1}, b_{i_2}, \dots\}$



not an anti-chain