10\7\11 Paul Erdös

Coloring Problem:

$A_1, A_2, \ldots, A_n \subseteq A$ and $|A_i| = k$, $1 \leq i \leq n$.

We want to color the elements of A, Red or Blue in such a way that no A_i is all one color. Can it be done?

Not always possible:

$n = 3$ $\qquad A_1 = \{1, 2\}$ $\qquad A = \{1, 2, 3\}$

$A_2 = \{1, 3\}$

$A_3 = \{2, 3\}$

$R = \{a \in A : a \text{ is Red}\}$ $\qquad |R| + |B| = 3$

$|R| \geq 2$ or

$B = \{a \in A : a \text{ is Blue}\}$ $\qquad |B| \geq 2$

If $|R| \geq 2$ then $R \supseteq A_1$ or A_2 or A_3

Theorem

If $n < 2^{k-1}$ then $\exists$ partition $R \cup B$

s.t.

$$R \cap A_i \neq \emptyset \quad 1 \le i \le n$$

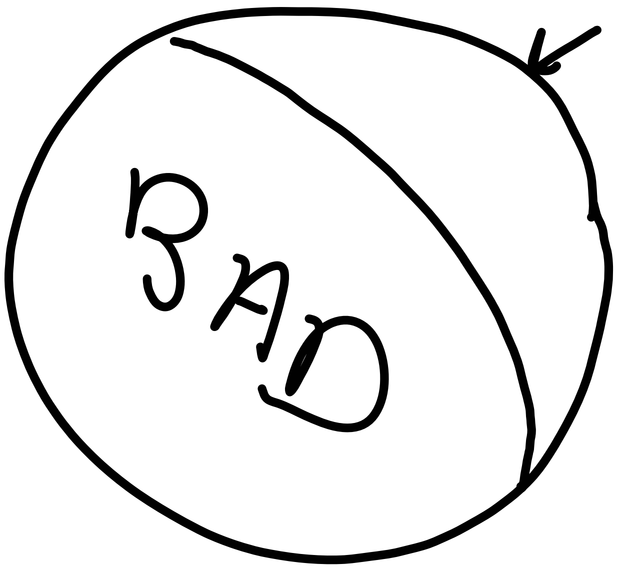
$$B \cap A_i \neq \emptyset \quad 1 \le i \le n$$

Proof

Take a random coloring of A.

BAD = event $\{\exists i : A_i \subseteq R \text{ or } A_i \subseteq B\}$

all colorings



We show

$Pr(BAD) < 1$

$\Longrightarrow \exists$ a coloring we want

$$BAD = \bigcup_{i=1}^{n} BAD(i)$$

A_i is bad

i.e. $A_i \subseteq R$ or

$A_i \subseteq B$

$$Pr(X_1 \cup X_2 \cup \dots) \le Pr(X_1) + Pr(X_2) + \dots$$

$$Pr(BAD) = Pr\left(\bigcup_{i=1}^{n} BAD(i)\right)$$

$$\le \sum_{i=1}^{n} Pr(BAD(i))$$

$|A_i| = k$

$A_i = \{x_1, x_2, \dots, x_k\}$

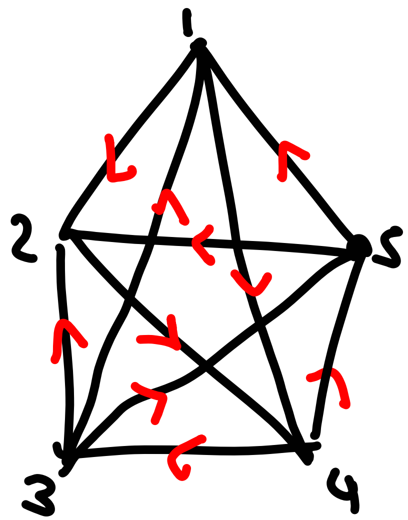
$BAD(i) = \left(Col(x_2) = Col(x_1)\right) \& \left((Col(x_3) = Col(x_1)\right) \dots$

$$= \sum_{i=1}^{n} \frac{1}{2^{k-1}}$$

< 1 by assumption.

Tournaments

A tournament is an orientation of a complete graph.



5 players : $5 \to 1$

Player 1 beat Player 5 in a game.

Who won?

Property B_k : for every set S, $|S| = k$
$\exists\, x \notin S$ such that
x beats everyone in S.

Do there exist tournaments with property k.

$\underline{k=1}$

$\underline{k=2}$

Assume n is large, with respect to k and choose a random tournament on n vertices:

we toss a fair coin to determine the outcome of each game.

$\mathcal{E}$ is the event that $\exists S, |S| = k$ and nobody beats all of S.

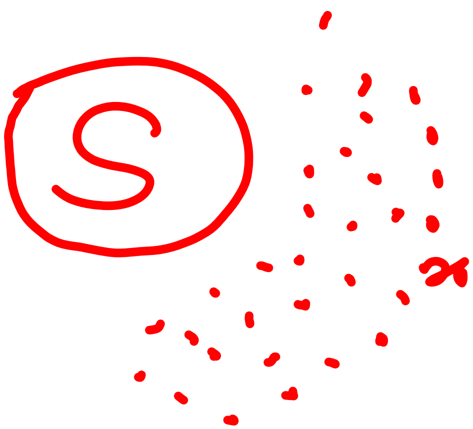
$$\mathcal{E} = \bigcup_S \mathcal{E}_S$$

$\mathcal{E}_S$: nobody beats everybody in S

$$\Pr(\mathcal{E}) < 1$$

$$\Pr(\mathcal{E}) = \Pr\left(\bigcup_S \mathcal{E}_S\right) \qquad |S| = k$$

$$\le \sum_S \Pr(\mathcal{E}_S) = \binom{n}{k}\left(1-\frac{1}{2^k}\right)^{n-k}$$



$\Pr(x \text{ beats } S) = \frac{1}{2^k}$

$\Pr(x \text{ doesn't beat } S) = 1 - \frac{1}{2^k}$

$\Pr(\mathcal{E}_S) = \left(1-\frac{1}{2^k}\right)^{n-k}$

$$\Pr(\mathcal{E}) \leq \binom{n}{k}\left(1-\frac{1}{2^k}\right)^{n-k} \qquad n \gg k$$

$$< n^k\left(1-\frac{1}{2^k}\right)^{n/2} \leq n^k e^{-n/2^{k+1}}$$

$1+x \leq e^x \; , \forall x$

$x \geq 0 \quad 1+x \leq 1+x+\frac{x^2}{2}+\cdots$

$x < -1 \quad 1+x < 0 \leq e^x$

$x = -y \qquad 1-y \leq e^{-y} = (1-y) + \underbrace{\left(\frac{y^2}{2} - \frac{y^3}{6}\right)}_{\geq 0} + \cdots$

$0 \leq y \leq 1$

$$\Pr(\mathcal{E}) < n^k e^{-n/2^{k+1}}$$

Fix k

Choose n such

$$\frac{n}{2^{k+1}} > k \log n$$

or

$$\frac{n}{\log n} > k 2^{k+1}$$

$$\Pr(\mathcal{E}) < n^k \times e^{-k \log n} = 1.$$