10/3/11

$$O_x = \{ y \in X : \exists g \in G : g * x = y \}$$

orbit

Claim: the orbits partition X.

i.e. they define an equivalence relation on X.

$x \sim y$ iff $\exists g \in G : g * x = y$

$\sim$ is an equivalence relation.

Orbits will be the equivalence classes of $\sim$

(a) $1_X * x = x \Rightarrow \sim$ is reflexive.

(b) $y = g * x \Rightarrow x = g^{-1} * y$ symmetric

$$g^{-1} * y = g^{-1} * (g * x)$$
$$= (g^{-1} \circ g) * x$$
$$= 1_X * x$$
$$= x$$

(c) $y = g * x$ $\qquad z = \underbrace{(h \circ g)}_{\in G} * x$ transitive

$z = h * y$

Claim: $|O_x| \cdot |S_x| = |G|$

$\boxed{\text{Orbit-Stabilizer theorem}}$

Stabilizer of x

$= \{ g \in G : g * x = x \}$

S_x is a sub-group of G.

$S_x \subseteq G$ and it is a group in its own right.

Check: $1 \in G$

& $g \in S_x$ & $h \in S_x$ then $(g \circ h) * x = g * (h * x) = x$

Define an equivalence relation on G.

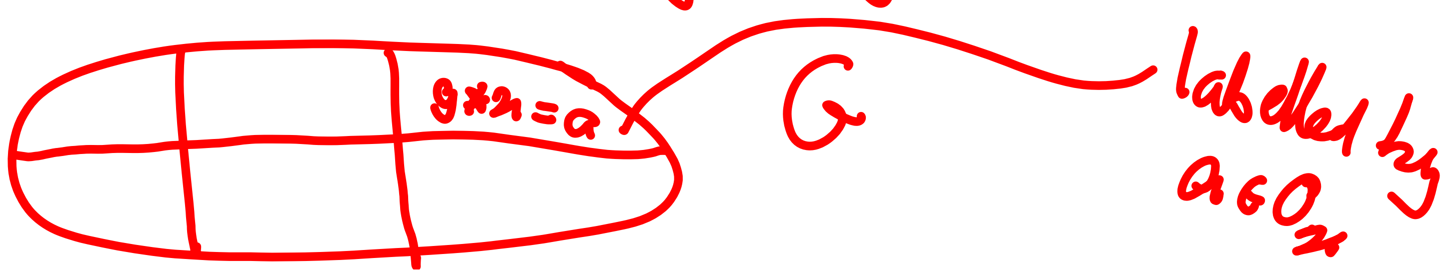
$g \sim h$ iff $g * x = h * x$

Clearly $\sim$ is an equivalence relation

(1) One equivalence class for every member of O_x

$\#$classes $= \#$ of possibilities for $g * x$, $g \in G$

Every class is determined by a $g * x$.



(11) $g \sim h$ iff $g * x = h * x$

iff $x = (g^{-1} \circ h) * x$

iff $g^{-1} \circ h \in S_x$

iff $h \in g \circ S_x$

$\# h$ = size of equivalence class of g

$= |g \circ S_x| = |S_x|$

$= |g \circ S_n| = |S_n|$??

Suppose $a, b \in S_n$ and $g \circ h_1 = g \circ h_2$

$$\Downarrow$$

$$h_1 = h_2$$

$$g \circ S_n = \{ g \circ h : h \in S_n \}$$

Example 2

b	b
r	r

$G = \{e, a, b, c, p, q, r, s\}$

$x = b\,b\,r\,r$

$O_x =$

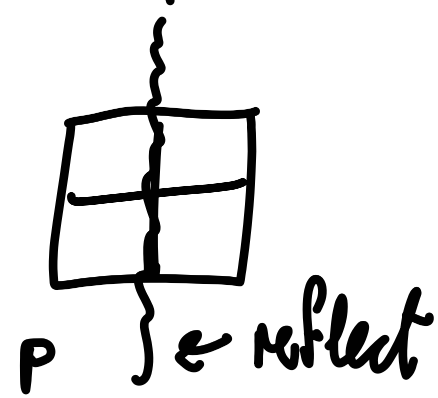
b	b
r	r

r	r
b	b

r	b
r	b

b	r
b	r

$S_x = \{e, p\}$



ν = # of orbits = what we want to know

$|O_x||S_x| = |G|$

each member of O_x contributes $1/|O_x|$ to the sum.

O_x contributes $|O_x| \times \frac{1}{|O_x|} = 1$ to the sum.

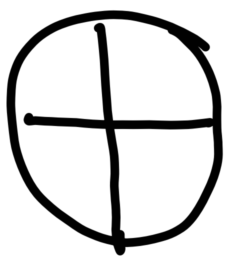
$$= \sum_{x \in X} \frac{1}{|O_x|}$$

$$= \frac{1}{|G|} \sum_{x \in X} |S_x|.$$

{1, 2, 3, 4} {5, 6} {8, 9, 10}

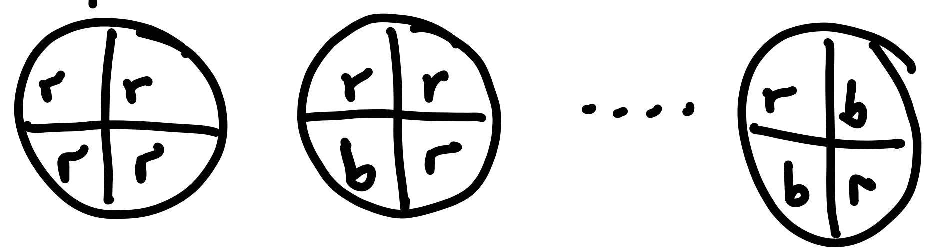
$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ $\frac{1}{2}, \frac{1}{2}$ $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

Example 1: $n=4$



4 rotations

$$\nu = \frac{1}{4}\left(4 + 1 + \cdots + 2 + \cdots\right) = 6$$



X

Tough to use if X is large.