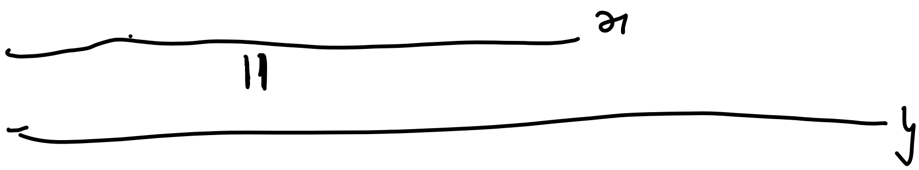
10\31\11

Kraft's Inequality

$x = x_1 x_2 \cdots x_m$ and $y = y_1 y_2 \cdots y_n$ are strings over some alphabet Σ. $|\Sigma| = s < \infty$

x is a <u>prefix</u> of y if $y_i = x_i$, $1 \le i \le m$



Suppose $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$ are strings over Σ and $|x^{(i)}| = n_i$ — $|x|$:= #symbols in x.

Assume that no $x^{(i)}$ is a prefix of any other $x^{(j)}$ — a bit like Sperner set up

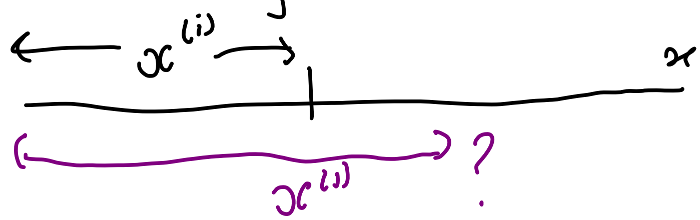
Then
$$\sum_{i=1}^{m} s^{-n_i} \leq 1$$

Proof

Let $n = \max\{n_1, n_2, \dots n_m\}$

Let x be a random string of length n over Σ.

Event $\mathcal{E}_i = \{x^{(i)}$ is a prefix of $x\}$



(a) $\Pr(\mathcal{E}_i) = s^{-n_i}$

(b) $\mathcal{E}_i$ and $\mathcal{E}_j$ are disjoint.

So $\Pr(\bigcup \mathcal{E}_i) = \sum_{i=1}^{m} \Pr(\mathcal{E}_i) \le 1$.

Pigeon Hole Principle (PHP)

Put m objects (pigeons) into n holes

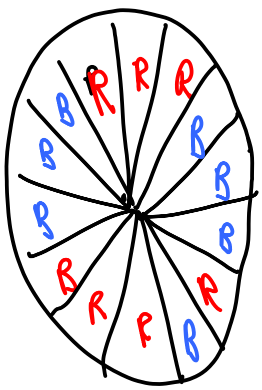
then at least one hole contains $\geq \left\lceil \frac{m}{n} \right\rceil$ objects.

Claim: $\exists \geq 10^4$ people in China with same number of strands of hair

Proof # people in China $\geq 10^9$. Now place person i into

hairs $0 \leq x \leq 10^5 - 1$ box i = # hairs

Some box must get $\geq 10^4$.

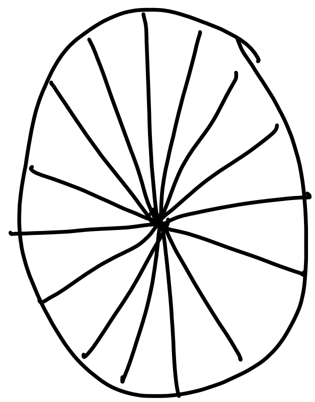


Disk 1

200 Sectors

100 Red

100 Blue



Disk 2

200 Sectors

Arbitrarily colored Red/Blue

Claim

It is possible to put Disk 2 onto the top of Disk 1 so that ≥ 100 sectors have same color below

Fix position Disk 1.

There are 200 ways of placing Disk 2.

q_i = # of coincidences for position i.

Claim

$$q_1 + q_2 + \cdots + q_{200} = 200 \times 100$$

$$\Rightarrow \exists i : q_i \geq 100$$

Define

$$A(i,j) = \begin{cases} 1 : & \text{sector } j \text{ is on top of same color in position } i \\ 0 : & \text{otherwise} \end{cases}$$

$$\begin{array}{cc} & j \\ i & \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \end{array} \leftarrow \text{row sum} = q_i$$

column sum = 100

Theorem (Erdös-Szekeres)

Let $a_1, a_2, \ldots, a_{k^2+1}$ be an arbitrary sequence of real numbers.

Then there must be a monotone subsequence of length $\geq k+1$

[Monotone increasing $a_{i_1} \leq a_{i_2} \leq \cdots$
decreasing $a_{i_1} \geq a_{i_2} \geq \cdots$

$k=3$

8 33 28 47 13 94 0 14 22 12

* is monotone increasing

E-S is best possible

$n, n-1, \dots, 1, 2n, 2n-1, \dots, n+1, \dots \dots n^2, n^2-1, \dots, n^2-n+1$

Longest monotone sequence is length n.