10\28\11

$\mathcal{P}_n = \{A : A \subseteq [n]\}$ power set

One is interested in the maximum size of a family of sets $\mathcal{A} \subseteq \mathcal{P}_n$ that have a given property: Extremal Set Theory

Example 1: Sperner Families

$\mathcal{A}$ is a Sperner family if $A, B \in \mathcal{A}$

sets

implies $A \nsubseteq B$ and $B \nsubseteq A$.

Sets are pairwise incomparable — anti-chain

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$$

This can be achieved by letting $\mathcal{A} = \{A : |A| = \lfloor n/2 \rfloor\}$

We will prove

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1 \qquad \text{LYM inequality}$$

$$\Downarrow$$

$$1 \geq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|\mathcal{A}|}{\binom{n}{\lfloor n/2 \rfloor}}$$

Let π be a random permutation $\{1,2,\dots,n\}$

$$\mathcal{E}_A : \underbrace{* \; * \; * \; * \; * \; * \; * \; *}_{\text{permutation } A} \;\Big|_{|A|}\; \underbrace{* \; * \; * \; * \; * \; * \; * \; *}_{\text{rest}}$$

$$\Pr(\mathcal{E}_A) = \frac{(|A|)!\,(n-|A|)!}{n!} = \frac{1}{\binom{n}{|A|}}$$

Because $\mathcal{A}$ is a Sperner family, these events are disjoint i.e $\mathcal{E}_A \cap \mathcal{E}_B = \emptyset$ for $A, B \in \mathcal{A}$

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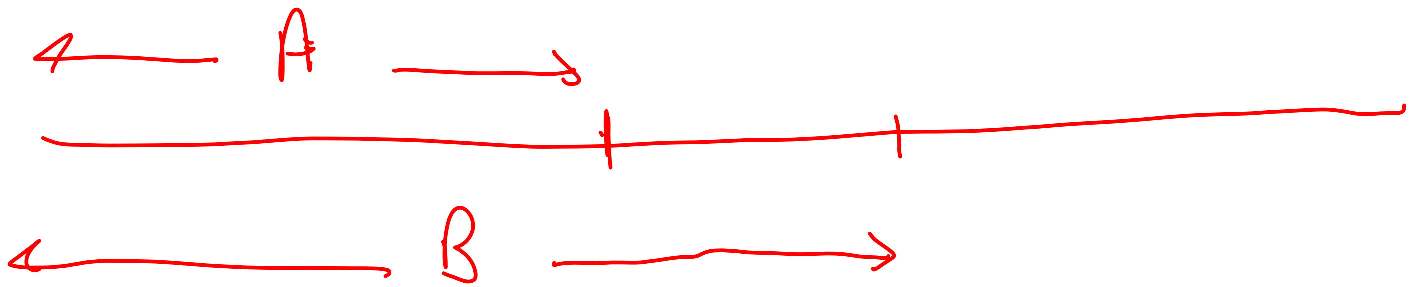


$$\Downarrow$$

$$1 \geq \Pr\Big(\bigcup_{A \in \mathcal{A}} \mathcal{E}_A\Big) = \sum_{A \in \mathcal{A}} \Pr(\mathcal{E}_A)$$

LYM

Intersecting Families

$\mathcal{A} \subseteq \mathcal{P}_n$ is said to be *intersecting*

if $A, B \in \mathcal{A} \Rightarrow A \cap B \neq \emptyset$

$|\mathcal{A}| \leq 2^{n-1}$

(1) Pair up each set $X \subseteq [n]$ with its complement $\overline{X}$

2^{n-1} pairs

An intersecting family can have at most one of $X, \overline{X}$

$\mathcal{A} = \{ A : 1 \in A \}$ then $\mathcal{A}$ is an intersecting family of size 2^{n-1}

Erdös-Ko-Rado Theorem

$\mathcal{A}$ is an intersecting family.

$A \in \mathcal{A} \Rightarrow |A| = k \leq \lfloor n/2 \rfloor$.

Then $|\mathcal{A}| \leq \binom{n-1}{k-1}$

$k > \lfloor n/2 \rfloor \Rightarrow$ one can take all k-sets and get

$|\mathcal{A}| = \binom{n}{k}$.

You can achieve $\binom{n-1}{k-1}$ by taking

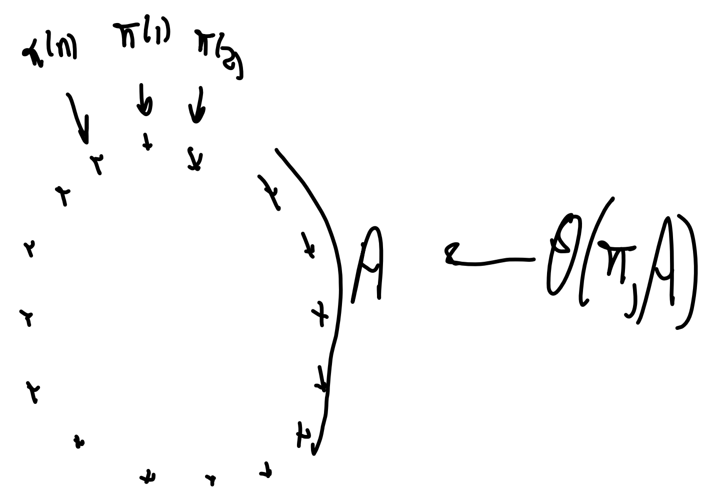
$$\mathcal{A} = \left\{ A \in \binom{[n]}{k} : 1 \in A \right\}$$

Proof of EKR

π is a permutation of $[n]$

$A \in \mathcal{A}$

$$\theta(\pi, A) := \begin{cases} 1 : & \exists s : A = \{\pi(s), \pi(s+1), \dots, \pi(s+k-1)\} \\ 0 : & \text{otherwise} \end{cases}$$



Key observation

$$\sum_{A \in \mathcal{A}} \theta(\pi, A) \le k$$

Suppose π is a random permutation:

$$\Pr(\theta(\pi, A) = 1) = E(\theta(\pi, A)) = n \cdot \frac{k!(n-k)!}{n!} = \frac{k}{\binom{n-1}{k-1}}$$

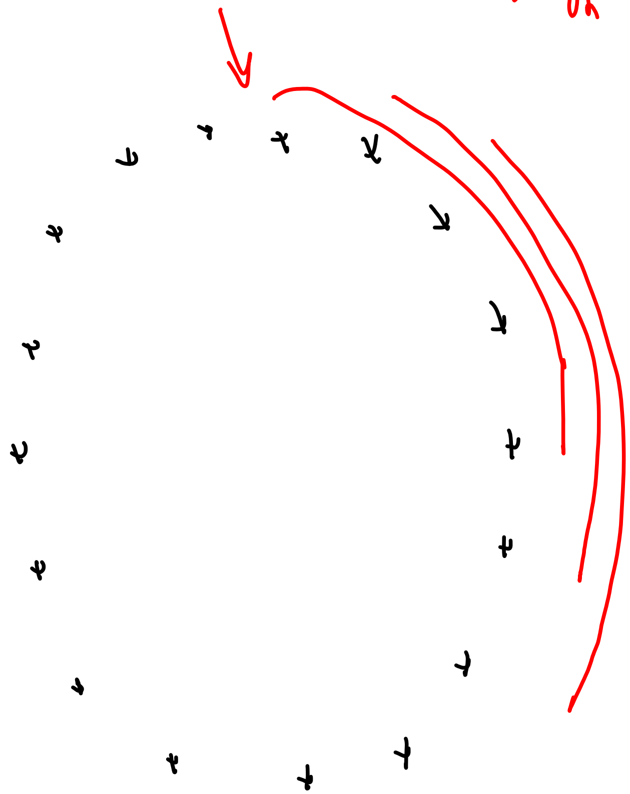
where A starts

$$E\Big(\sum_{A \in \mathcal{A}} \theta(\pi, A)\Big) = |\mathcal{A}| \cdot \frac{k}{\binom{n-1}{k-1}}$$

$\leq k$

$\Rightarrow$ EKR

left most of collection



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