Hall's Theorem

G is bipartite with vertex partition A, B

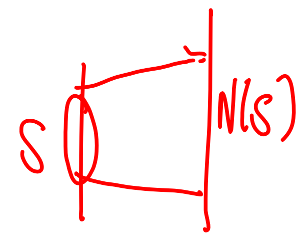
Theorem

$\exists$ a matching M of size $|A|$ iff

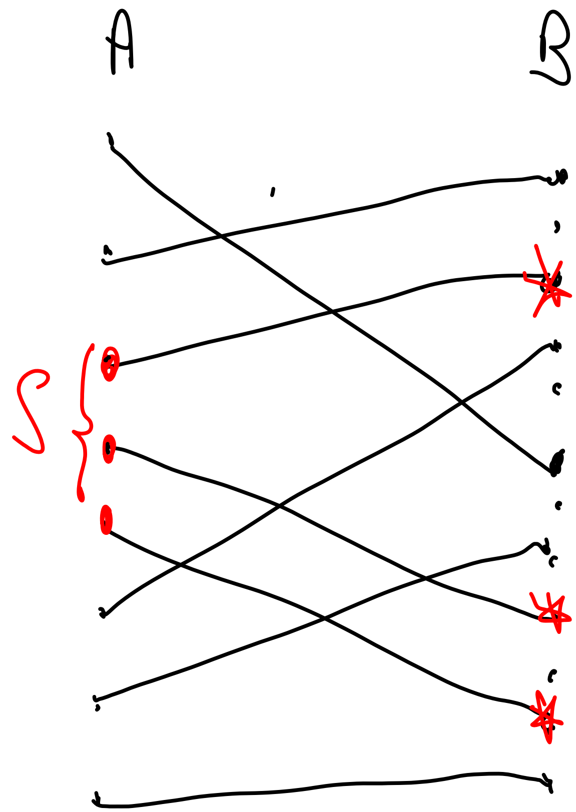
→ Hall's condition

$$|N(S)| \geq |S| \quad \text{for all } S \subseteq A.$$

nbrs of S in B



Suppose there is a matching of size $|A|$



$M := \{(a, \underbrace{f(a)}_{\in B})\}$

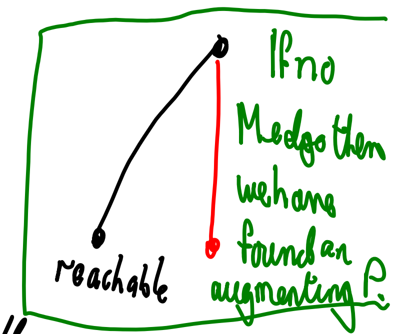
$S := \{s_1, s_2, \dots, s_k\}$

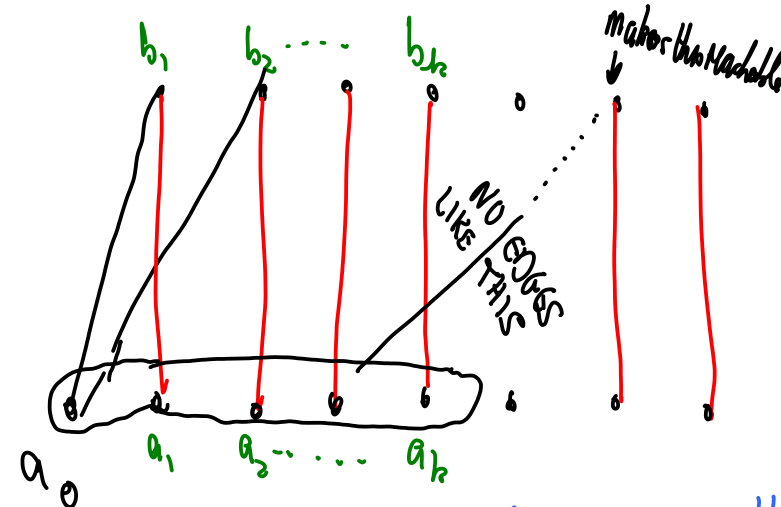
then

$N(S) \supseteq \{\underbrace{f(s_1), f(s_2), \dots, f(s_k)}_{\text{distinct}}\}$

Suppose now that the maximum matching M has $|M| < |A|$.

Choose some unsaturated $a_0 \in A$.





a_i & b_i are reachable from a_0 by alternating paths

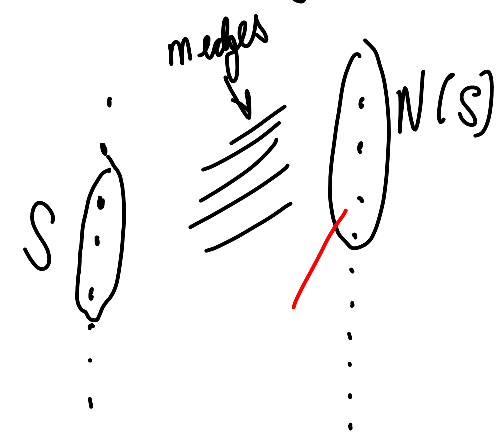
$S := \{a_0, a_1, \dots, a_k\}$ $N(S) := \{b_1, b_2, \dots, b_k\}$

$\square$

Marriage Theorem

$k|A| = k|B| = \#\text{edges}$

If $G = (A \cup B, E)$ is k-regular ($k \geq 1$) (i.e. degree of every vertex is exactly k) then G has a <u>perfect</u> matching (one that saturates every vertex.



$m = k|S|$

$m \leq k|N(S)|$

$\Downarrow$

$|N(S)| \geq |S|$.

Corollary

If G is a bipartite k-regular graph $(G=(A\cup B, E))$

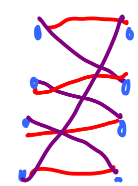
then we can decompose

$$E = M_1 \cup M_2 \cup \cdots \cup M_k$$

Proof by induction on k.
M_k exists (marriage thm)
$G - M_k$ is $(k-1)$ regular

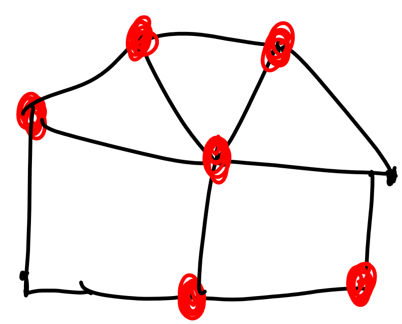
where the M_i are disjoint perfect matchings

So if you give M_i color i then the edges are colored so that no two adjacent edges have the same color:



König's Theorem - Edge Covers.

$G = (V, E)$. $S \subseteq V$ is an edge cover if every edge of G contains a vertex in S.



If S is a cover and M is a matching then

$|S| \geq |M|$.

Need a vertex for every $e \in M$.

$\max |M| \leq \min |S|$

M is a matching, S is an edge cover

König's Theorem: If G is bipartite then

$\max |M| = \min |S|$.