10\12\11

Secretary Problem.

Pat wants to hire a new secretary.

She interviews $P_1, P_2, \ldots, P_n$ in random order of suitability.

Strategy: fix some $m < n$.
interview $P_1, P_2, \ldots, P_m$ to evaluate the market.
Then interview $P_{m+1}, \ldots P_n$ and stop when she sees the best one so far.

She may not hire anybody.

Question: what is the probability that she hires the best person?

$S = \{$ she succeeds in hiring the best person $\}$.

$$\Pr(S) = \sum_{i=1}^{n} \Pr(S \mid P_i) \Pr(P_i) = \frac{1}{n} \sum_{i=1}^{n} \Pr(S \mid P_i)$$

total law of probability

P_i: i th person is the best.

$$\Pr(S \mid P_i) = 0 \quad \text{if } i \leq m$$

Given P_i, she succeeds iff the best of the best person in $1, 2, \cdots, i-1$ occurs among the first m.

So $$\Pr(S \mid P_i) = \frac{m}{i-1}$$

$$\Pr(S) = \frac{m}{n} \sum_{i=m+1}^{n} \frac{1}{i-1}$$

$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \log_e n + \gamma_n$

harmonic numbers

γ constant

Euler's Constant

$H_n \sim \int_{x=1}^{n} \frac{dx}{x}$

Suppose n is large and $m = \alpha n$.

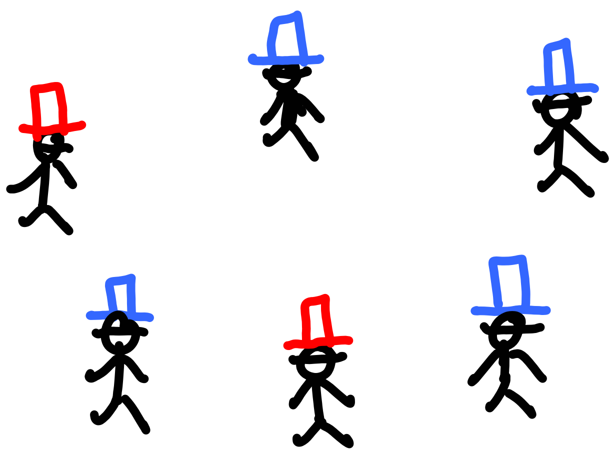
We optimize over α.

$$\Pr(S) \sim \alpha(\log n - \log \alpha n)$$

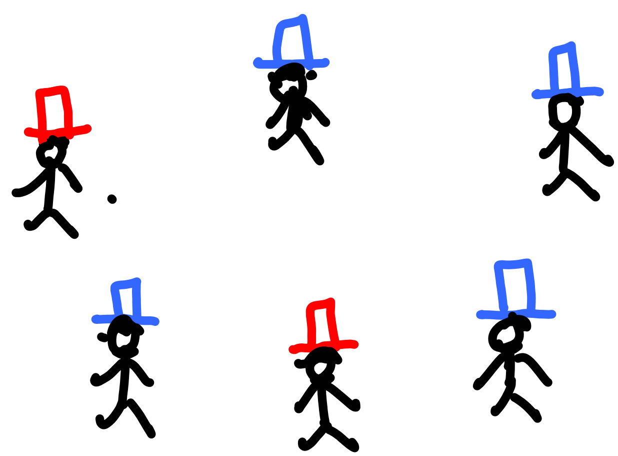
$$= \alpha \log \frac{1}{\alpha}$$

$\alpha = 1/e$ maximises this. $\Pr(S) \sim 1/e$.

A problem with hats



take off blindfolds



n people.

blindfolded

Red\Blue hats

are placed

randomly on

their heads.

Everybody sees

everyones hat

but their own.

A person can either

(i) Say nothing

or

(ii) guess the color of their hat.

Outcome:

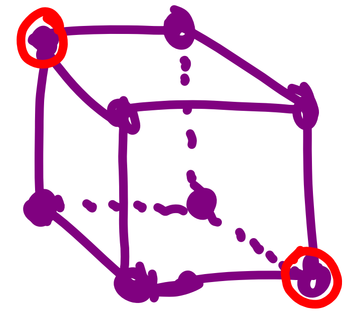
If everyone who guesses, guesses right — big prize

If somebody guesses wrong — big punishment.

Pr(prize) ?

Can find a strategy such $Pr(\text{prize}) = 1 - O\left(\frac{\log n}{n}\right)$

We partition $Q_n = \{0,1\}^n$ into 2 sets W, L.



L is a cover if $x \in Q_n \setminus L$ implies there exists y L such that

$$h(x,y) = |\{ i : x_i \neq y_i \}| = 1$$

hamming distance

We show there is a cover of size $\leq \frac{2\ln n}{n} \times 2^n$.

Hat people agree on a cover L of this size.

They agree to assume that the hat (Red = 0, Blue = 1)

define $x \notin L$: $\Pr(x \notin L) \geq 1 - \frac{2\ln n}{n}$

2 possibilities for x. — x', x''

If both in L or both in W – say nothing.

x' in L, x'' in W assume color give x''.

Small cover:

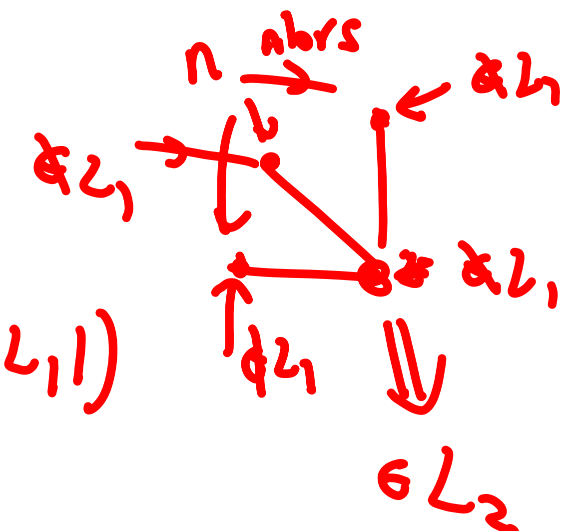
$$\boxed{1-p \leq e^{-p}}$$

$$p = \frac{\ln n}{n}$$

Randomly put $x \in Q_n \longrightarrow L_1$ with probability p.

(L_1 need not be a cover).

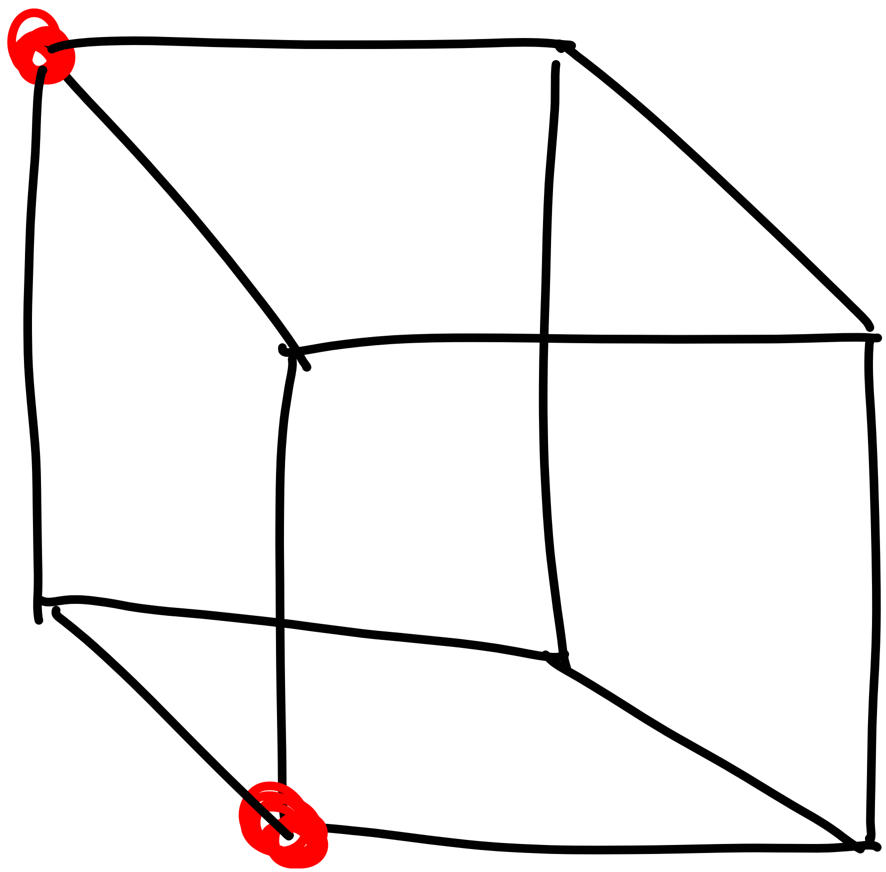
$L_2 = \{x$ not covered by $L_1\}$



$L = L_1 \cup L_2$ is a cover.

$$E(|L|) = \underbrace{2^n p}_{E(|L_1|)} + 2^n (1-p)^{n+1} \leq 2^n \cdot \frac{2 \ln n}{n}$$

$E(1L1)$



L_1

L_2

Vertex = { coloring of the halo }

$$1+x \leq e^{x}, \forall x$$

$$E(|L|) = 2^n p + 2^n (1-p)^{n+1}$$

$$\leq 2^n p + 2^n e^{-(n+1)p}$$

$$\leq 2^n p + 2^n e^{-np}$$

$$= 2^n \frac{\log n}{n} + 2^n/n \leq 2^n \cdot \frac{2\log n}{n} \qquad np = \log n$$

$$\Rightarrow \exists L : |L| \leq 2^n \cdot \frac{2\log n}{n}.$$

First Moment Method (Markov Inequality)

X is a random variable taking value in $\{0, 1, 2, \ldots\}$

$$\Pr[X \geq 1] \leq E(X)$$

$$E(X) = E(X \mid X=0)\Pr(X=0) + E(X \mid X \geq 1)\Pr(X \geq 1)$$
$$\geq \Pr(X \geq 1)$$

Intersection Safe Familier

$\mathcal{A}$ is a collection of subsets of $[n]$

$\mathcal{A}$ is intersection safe if

$\nexists$ distinct $A, B, C \in \mathcal{A}$ s.t.

$$A \cap B \subseteq C$$

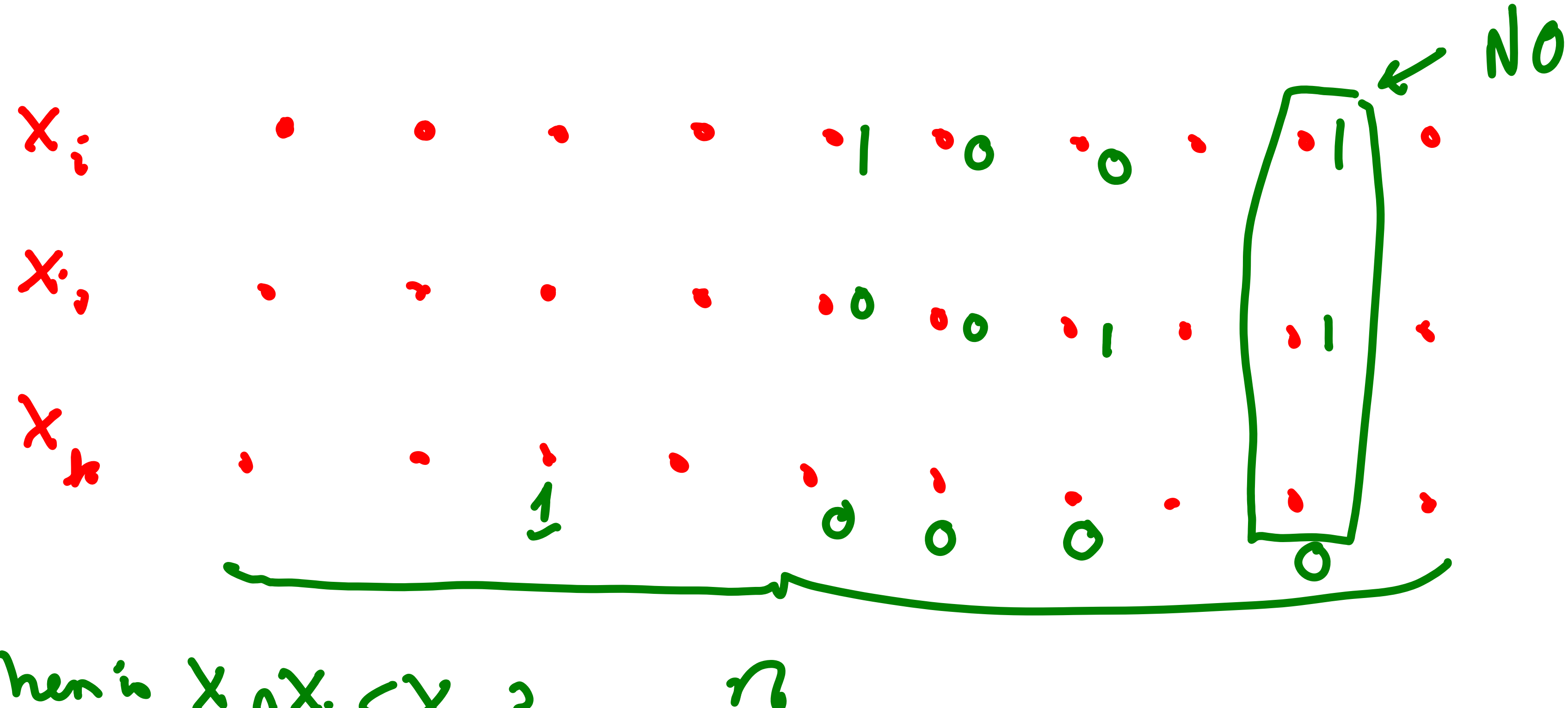
Let $\mathcal{A} = \{X_1, X_2, \ldots\ldots, X_p\}$ is a collection of p independently and randomly chosen sets.

$$Z = \{(i, j, k) : X_i \cap X_j \subseteq X_k\}$$

$$\Pr(Z \geq 1) \leq E(Z)$$ ← if this is < 1 then a family of size p exists

$$E(Z) = p(p-1)(p-2)\Pr(X_i \cap X_j \subseteq X_k)$$

$$E(Z) = p(p-1)(p-2)\underbrace{\Pr(X_i \cap X_j \subseteq X_k)} < p^3\left(\frac{7}{8}\right)^n$$



When is $X_i \cap X_j \subseteq X_k$?

So if $p^3 \left(\frac{7}{8}\right)^n \leq 1$ then $\exists$ an intersection safe family

$p \geq \left(\frac{8}{7}\right)^{n/3}$ is O.K.

n people

Everybody knows the indices & the keys given to everyone else.