10\12\11

Secretary Problem.

Pat wants to hire a new secretary.

She interviews $P_1, P_2, \ldots, P_n$ in random order of suitability.

Strategy: fix some $m < n$.
interview $P_1, P_2, \ldots, P_m$ to evaluate the market.
Then interview $P_{m+1}, \ldots, P_n$ and stop when she sees the best one so far.

She may not hire anybody.

Question: what is the probability that she hires the best person?

$S = \{$ she succeeds in hiring the best person $\}$.

$$\Pr(S) = \sum_{i=1}^{n} \Pr(S \mid P_i)\Pr(P_i) = \frac{1}{n}\sum_{i=1}^{n} \Pr(S \mid P_i)$$

total law of probability

i th person is the best.

$Pr(S \mid P_i) = 0$ if $i \le m$

Given P_i, she succeeds iff the best of the best person in $1, 2, \cdots, i-1$ occurs among the first m.

So $Pr(S \mid P_i) = \frac{m}{i-1}$

$$\Pr(S) = \frac{m}{n} \sum_{i=m+1}^{n} \frac{1}{i-1}$$

Suppose n is large and $m = \alpha n$.

We optimize over α.

$$\Pr(S) \sim \alpha(\log n - \log \alpha n)$$

$$= \alpha \log \frac{1}{\alpha}$$

$\alpha = 1/e$ maximizes this. $\Pr(S) \sim 1/e$.

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \log_e n + \gamma_n$$

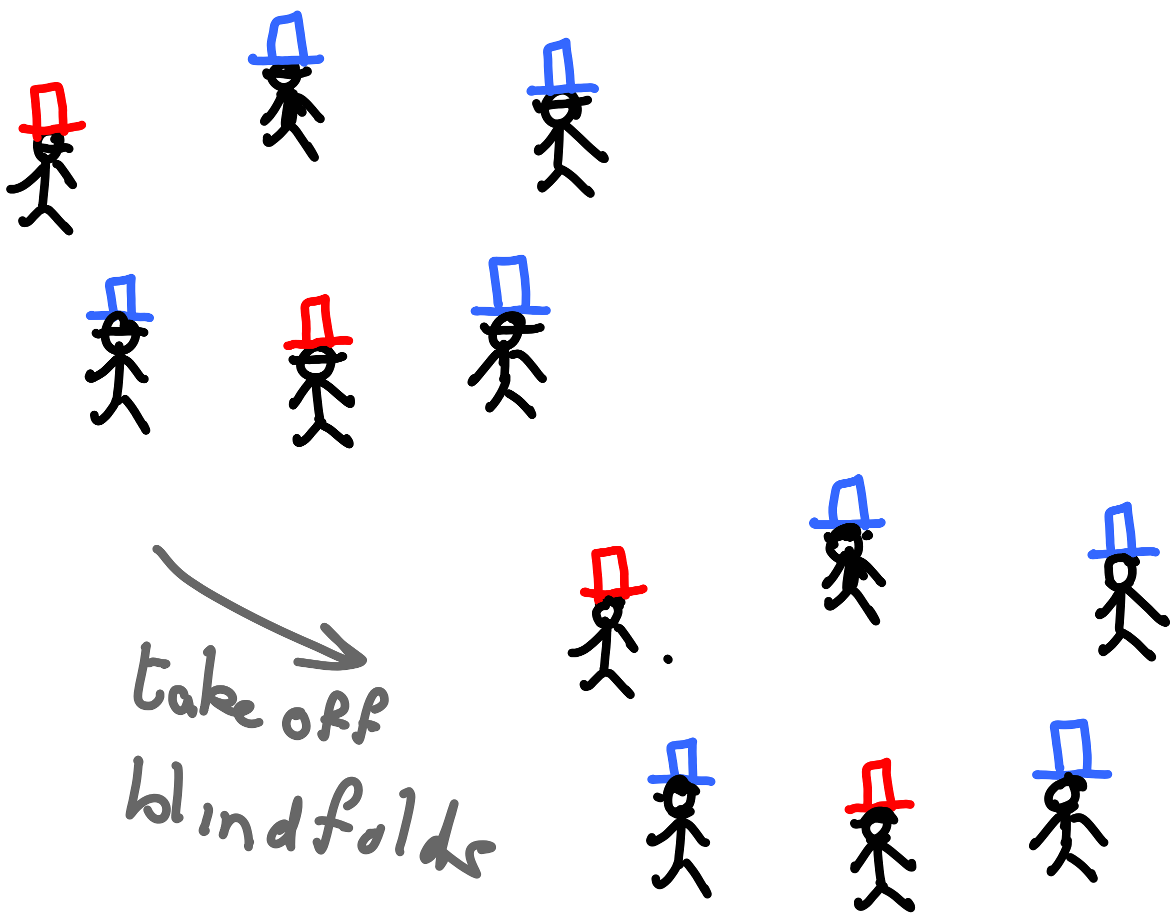
harmonic numbers

γ constant

Euler's constant

$$H_n \sim \int_{x=1}^{n} \frac{dx}{x}$$

A problem with hats



n people.
blindfolded
Red\Blue hats
are placed
randomly on
their heads.

Everybody sees
everyones hat
but their own.

A person can either

(i) Say nothing

or

(ii) guess the color of their hat.

Outcome :

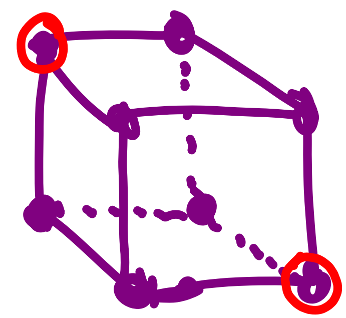
If everyone who guesses, guesses right - big prize

If somebody guesses wrong — big punishment.

$Pr(\text{prize})$?

Can find a strategy such $Pr(\text{prize}) = 1 - O\left(\frac{\log n}{n}\right)$

We partition $Q_n = \{0,1\}^n$ into 2 sets W, L.



Q_3

L is a <u>cover</u> if $x \in Q_n \setminus L$ implies there exists y L such that

$$h(x,y) = |\{i : x_i \neq y_i\}| = 1$$

hamming distance

We show there is a cover of size $\leq \frac{2\ln n}{n} \times 2^n$.

Hat people agree on a cover L of this size.

They agree to assume that the hat (Red $= 0$, Blue $= 1$)

define $x \notin L$: $\Pr(x \notin L) \geq 1 - \frac{2\ln n}{n}$

2 possibilities for x — x', x''

If both in L or both in W - say nothing.

x' in L, x'' in W assume color gives x''.

Small cover:

$p = \frac{\ln n}{n}$

Randomly put $x \in Q_n \longrightarrow L_1$ with probability p.

(L_1 need not be a cover).

$L_2 = \{x$ not covered by $L_1\}$

$L = L_1 \cup L_2$ is a cover.

$E(|L|) = 2^n p + 2^n (1-p)^{n+1}$