10\10\11

We will color the elements of A one at a time, in such a way that if $n < 2^{k-1}$ then we end up with a good coloring

Suppose at some stage we have colored the elements $C \subseteq A$

$= R \cup B$

elements colored Red so far

elements colored Blue so far

Define a random variable

$Z(R,B) = \#$ of badly colored sets in a random completion of C.

$$E(Z(R,B)) = \sum_{i=1}^{n} Z_i(R,B)$$ ← probability

← indicate whether or not A_i is mono-colored

$$Z_i(R,B) = \begin{cases} 1 & A_i \subseteq R \text{ or } A_i \subseteq B \\ 0 & A_i \cap R \neq \emptyset \text{ and } A_i \cap B \neq \emptyset \\ 2^{1-k} & A_i \cap C = \emptyset \\ 2^{-|A_i \setminus R|} & A_i \cap R \neq \emptyset \text{ \& } A_i \cap B = \emptyset \\ 2^{-|A_i \setminus B|} & A_i \cap R = \emptyset,\ A_i \cap B \neq \emptyset \end{cases}$$

Initially $E(Z(\emptyset,\emptyset)) < 1$

Choose any $x \notin C$.

Suppose we gave it a random color c.

Suppose $E(Z(R,B)) < 1$

$$1 > E(Z(R,B))$$
$$= E(Z(R,B) \mid c = \text{Red}) \Pr(c = \text{Red})$$
$$+ E(Z(R,B) \mid c = \text{Blue}) \Pr(c = \text{Blue})$$
$$= \frac{E(Z(R \cup x, B)) + E(Z(R, B \cup x))}{2}$$

at least one of these is less than one

Choose a color for x such that the corresponding expectation is < 1.

Continuing in this way, we end up with

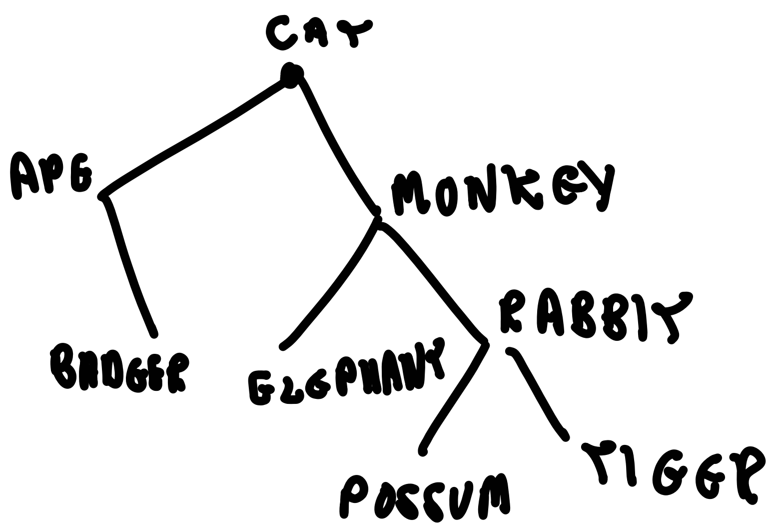
$R \cup B = A$

$E(Z(R,B)) < 1 \Rightarrow$ Coloring is good.

no randomness, this is just the number of badly colored sets.

Probabilistic Analysis of a Binary Search

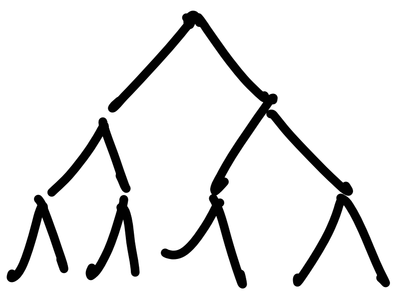
Tree



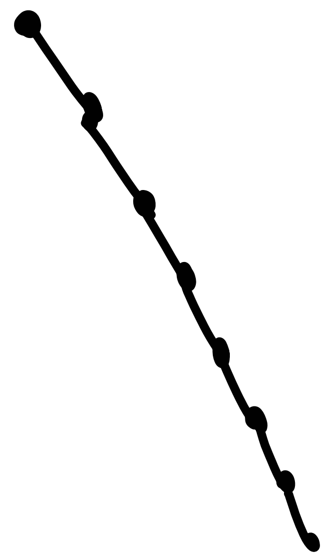
Maximum number of queries to check if an animal is in the tree = depth of the tree

Important to keep depth "small".

Depth



$\geq \log_2 n$



$\leq n$

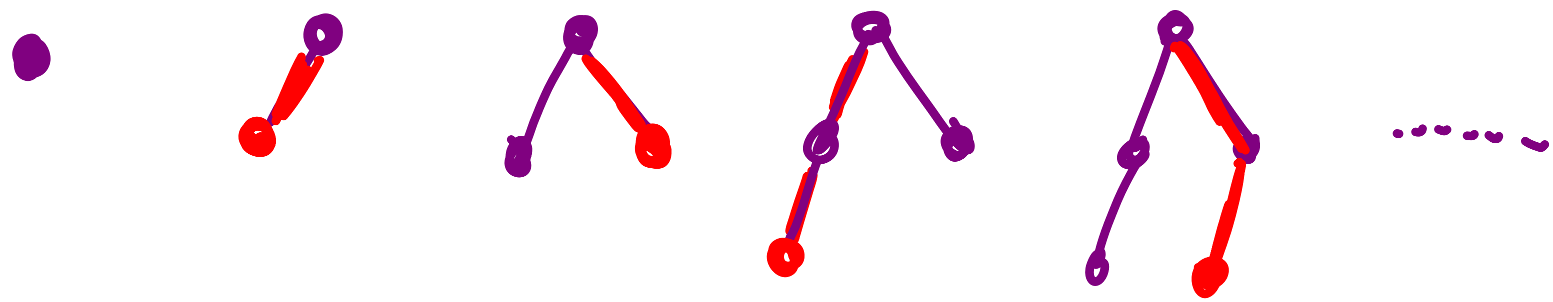
Nice algorithms for re-balancing to keep trees depth $O(\log n)$

We will assume a random model for keys entering the tree and show that w.h.p. no re-balancing is needed.
with high probability

Model

Start with $T_0 = \bullet^r$.

n'th particle enters tree. When it reaches an occupied node. It flips a coin and goes left or right. Settle in an empty place.



Claim: D_n = depth of tree after n insertions

$$\Pr(D_n \geq t) \leq \left(n\,2^{-(t-1)/2}\right)^t = \text{small if } t \gg 2\log_2 n$$

Coin flips: H T H H T H T T T H H T T H T H T H T T

* * * * * *

$$\text{DEEP} = \{D_n \geq t\}$$

$$= \bigcup_{\text{Paths } P} \; \bigcup_{\text{flips } S} \text{DEEP}(P, S)$$

$|P| = t \qquad S = \{s_1, s_2, \dots, s_t\}$

Particles $s_1, s_2, \dots, s_t$ create the path P in the lines

$$\Pr(\mathrm{DEEP}(P, S)) \le \frac{1}{2} \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^4 \times \cdots \times \left(\frac{1}{2}\right)^t$$

$$= \left(\frac{1}{2}\right)^{t(t+1)/2}$$

$$\Pr(\mathrm{DEEP}) \le \sum_{S} \sum_{P} \left(\frac{1}{2}\right)^{t(t+1)/2}$$

$$\#S = \binom{n}{t} \qquad \#P \le 2^t$$