21-301 Combinatorics
Homework 8
Due: Monday, November 1

1. Let $\mathcal{A}$ be an intersecting family of subsets of $[n]$ such that $A \in \mathcal{A}$ implies $k \leq |A| \leq \ell \leq n/2$. Show that

$$|\mathcal{A}| \leq \sum_{i=k}^{\ell} \binom{n-1}{i-1}.$$

Solution: Let $\mathcal{A}_i = \{A \in \mathcal{A} : |A| = i\}$. Then $\mathcal{A}_i$ is an intersecting family and so by the Erdős-Ko-Rado theorem, we have $|\mathcal{A}_i| \leq \binom{n-1}{i-1}$ and the result follows from $|\mathcal{A}| = |\mathcal{A}_k| + \cdots + |\mathcal{A}_\ell|$.

2. Let $m = \lfloor n/2 \rfloor$. Describe a family $\mathcal{A}$ of size $2^{n-1} + \binom{n-1}{m-1}$ that has the following property: If $A_1, A_2 \in \mathcal{A}$ are disjoint then $A_1 \cup A_2 = [n]$.

Solution: If $n = 2m+1$ is odd, let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ where $\mathcal{A}_1 = \{A \subseteq [n] : \ |A| \geq m+1\}$ and $\mathcal{A}_2 = \{A \subseteq [n] : \ |A| = m, \ A \ni 1\}$. Here $|\mathcal{A}_1| = 2^{n-1}$, because if we partition the subsets of $[n]$ into 2^{n-1} pairs, a set and its complement, then the larger of the two sets is in $\mathcal{A}_1$. Clearly $|\mathcal{A}_2| = \binom{n-1}{m-1}$. Both $\mathcal{A}_1$ and $\mathcal{A}_2$ are intersecting families and if $A \in \mathcal{A}_1, B \in \mathcal{A}_2$ and $A \cap B = \emptyset$ then we have $|B| = m$ and $|A| \geq m+1$ and so A, B must be complementary.

If $n = 2m$ is even then let $\mathcal{A} = \{A \subseteq [n] : \ |A| \geq m\}$. Now

$$|\mathcal{A}| = |\mathcal{A}_1| + \binom{n}{m} = |\mathcal{A}_1| + \frac{1}{2}\binom{n}{m} + \frac{1}{2}\binom{n}{m} = 2^{n-1} + \frac{1}{2}\binom{n}{m} = 2^{n-1} + \binom{n-1}{m-1}.$$

Here, $|\mathcal{A}_1| + \frac{1}{2}\binom{n}{m} = 2^{n-1}$ because we can obtain this number of sets by taking one set from each pair of complementary sets. Each $A \in \mathcal{A}_1$ intersects all sets of size m or more and two sets of size m fail to intersect only when they are complementary.

3. Subsets $A_i, B_i \subseteq [n]$, $i = 1, 2, \ldots, m$ satisfy $A_i \cap B_i = \emptyset$ for all i and $A_i \cap B_j \neq \emptyset$ for all $i \neq j$. Show that

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

Solution: Let π be a random permutation of $[n]$ and for disjoint sets A, B define the event $\mathcal{E}(A, B)$ by

$$\mathcal{E}(A, B) = \{\pi : \ \max\{\pi(a) : \ a \in A\} < \min\{\pi(b) : \ b \in B\}\}.$$

The events $\mathcal{E}_i = \mathcal{E}(A_i, B_i)$, $i = 1, 2, \ldots, m$ are disjoint. Indeed, suppose that $\mathcal{E}(A_i, B_i)$ and $\mathcal{E}(A_j, B_j)$ occur. Let $x \in A_i \cap B_j$ and $y \in A_j \cap B_i$. x, y exist by (ii) and (i) implies that they are distinct. Then $\mathcal{E}(A_i, B_i)$ implies that $\pi(x) < \pi(y)$ and $\mathcal{E}(A_j, B_j)$ implies that $\pi(x) > \pi(y)$, contradiction.

Observe next that for two fixed disjoint sets A, B, $|A| = a, |B| = b$ there are exactly $\binom{n}{a+b} a!b!(n-a-b)!$ permutations that produce the event $\mathcal{E}(A, B)$. Indeed, there are

$\binom{n}{a+b}$ places to position $A \cup B$. Then there are $a!b!$ that place A as the first a of these $a+b$ places. Finally, there are $(n-a-b)!$ ways of ordering the remaining elements not in $A \cup B$.

Thus

$$\begin{aligned} \Pr(\mathcal{E}(A_i, B_i)) &= \frac{n!}{(|A_i|+|B_i|)!(n-|A_i|-|B_i|)!}|A_i|!|B_i|!(n-|A_i|-|B_i|)!\frac{1}{n!} \\ &= = \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}}. \end{aligned}$$

But then the disjointness of the collection of events $\mathcal{E}(A_i, B_i)$ implies that

$$\sum_{i=1}^{m} \Pr(\mathcal{E}(A_i, B_i)) \leq 1.$$