21-301 Combinatorics
Homework 6
Due: Monday, October 11

1. Let $\chi(G)$ be the chromatic number of graph $G=(V,E)$. Let $\alpha(G),\kappa(G)$ denote the size of the largest independent set of G, clique of G respectively.

   Show that
$$\chi(G)\geq\max\left\{\frac{|V|}{\alpha(G)},\kappa(G)\right\}.$$
   Show further that $\chi(G)\chi(\bar{G})\geq|V|$.

   Here $\bar{G}$ is the complement of G i.e. the graph with edge set $\binom{V}{2}\setminus E$.

   Solution: If S is a clique of size s then we need at least s colors, just to color S. Thus $\chi(G)\geq\kappa(G)$. Observe next that in a proper coloring, a set of vertices of the same color must form an independent set. Thus a proper coloring partitions the vertex set into a set of color classes $C_1,C_2,\ldots,C_k$ where $|C_i|\leq\alpha(G)$ for $1\leq i\leq k$. This implies that $k\geq|V|/\alpha(G)$.

   For the second part, we have
$$\chi(G)\geq\frac{|V|}{\alpha(G)}=\frac{|V|}{\kappa(\bar{G})}.$$
   Hence,
$$\chi(G)\chi(\bar{G})\geq\chi(G)\kappa(\bar{G})\geq\frac{|V|}{\kappa(\bar{G})}\kappa(\bar{G})=|V|.$$

2. Let $G=(V,E)$ be a graph with kn vertices. Show, by the probabilistic method, that there is a partition $V=V_1\cup V_2\cup\cdots\cup V_k$ with $|V_i|=n$, $i=1,2,\ldots,k$ such that at most $|E|/k$ of the edges of G have both of their endpoints in the same part of the partition.

   Solution: Let $V_1,V_2,\ldots,V_k$ be a random partition of the vertex set. Let $e=(v,w)$ be an edge of E. Then
$$\Pr(\exists i:\ e\subseteq V_i)=\sum_{i=1}^{k}\Pr(w\in V_i\mid v\in V_i)\Pr(v\in V_i)=$$
$$\sum_{i=1}^{k}\frac{\binom{kn-2}{n-2}}{\binom{kn-1}{n-1}}\frac{1}{k}=\frac{n-1}{kn-1}<\frac{1}{k}.$$
   If Z is the number of edges of G that have both of their endpoints in the same part of the partition, then $\mathbf{E}(Z)\leq|E|/k$ and so the required partition must exist.

3. Let $G=(V,E)$ be an r-regular graph with n vertices i.e. every vertex has degree r. $S\subseteq V$ is a *dominating set* if $w\notin S$ implies that there exists $v\in S$ for which $\{v,w\}\in E$. Show, by the probabilistic method, that G has a dominating set of size at most $\frac{1+\ln r}{r}n$.

   Solution: Let $p=\frac{\ln r}{r}$ and let S_1 be a random sub-set of V where each $v\in V$ is placed inot S, independently with probability p. Let S_2 be the set of vertices that are not adjacent to any vertex of S_1. The set $S=S_1\cup S_2$ is a dominating set.

$$\mathbf{E}(|S|) = \mathbf{E}(|S_1|) + \mathbf{E}(|S_2|) = np + n(1-p)^{r+1} \leq np + ne^{-rp} \leq \frac{1+\ln r}{r} n.$$

So there must be a dominating set of the required size.