21-301 Combinatorics
Homework 6
Due: Monday, October 11

1. Let $\chi(G)$ be the chromatic number of graph $G = (V, E)$. Let $\alpha(G), \kappa(G)$ denote the size of the largest independent set of G, clique of G respectively.

   Show that
   $$\chi(G) \geq \max\left\{\frac{|V|}{\alpha(G)}, \kappa(G)\right\}.$$
   Show further that $\chi(G)\chi(\bar{G}) \geq |V|$.

   Here $\bar{G}$ is the complement of G i.e. the graph with edge set $\binom{V}{2} \setminus E$.

2. Let $G = (V, E)$ be a graph with kn vertices. Show, by the probabilistic method, that there is a partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ with $|V_i| = n$, $i = 1, 2, \ldots, k$ such that at most $|E|/k$ of the edges of G have both of their endpoints in the same part of the partition.

3. Let $G = (V, E)$ be an r-regular graph with n vertices i.e. every vertex has degree r. $S \subseteq V$ is a *dominating set* if $w \notin S$ implies that there exists $v \in S$ for which $\{v, w\} \in E$. Show, by the probabilistic method, that G has a dominating set of size at most $\frac{1+\ln r}{r}n$.