21-301 Combinatorics
Homework 4
Due: Monday, September 27

1. Suppose that you are asked to multiply a collection of $m \times m$ matrices to form the product $A_1 A_2 \cdots A_{n+1}$. Let $C_0 = 1$ and let C_n be the number of ways to do this. For example $C_2 = 2$. We can compute $(A_1A_2)A_3$ or $A_1(A_2A_3)$. Show that

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

Determine C_n.

Solution: The term $C_k C_{n-k}$ counts the number of ways of first computing the products $C_1 C_2 \cdots C_k$ and $C_{k+1} C_{k+2} \cdots C_{n+1}$ and then multiplying the two resulting matrices together. By summing over k we count all possible ways of doing the multiplication.

Let a_n be the number of solutions to the polygon triangulation problem. Thus $a_0 = 0, a_1 = a_2 = 1$ and $a_n = \sum_{k=0}^{n} a_k a_{n-k}$ for $n \geq 2$. We claim that $C_n = a_{n+1} = \frac{1}{n+1}\binom{2n}{n}$. We prove this by induction on n. It is clearly true for $n = 0$. So,

$$\begin{aligned}
C_{n+1} &= \sum_{k=0}^{n} C_k C_{n-k} \\
&= \sum_{k=0}^{n} a_{k+1} a_{n+1-k} \qquad \textit{by induction} \\
&= a_1 a_{n+1} + a_2 a_n + \cdots + a_{n+1} a_1 \qquad \textit{expanding sum} \\
&= a_0 a_{n+2} + a_1 a_{n+1} + a_2 a_n + \cdots + a_{n+1} a_1 + a_{n+2} a_0 \qquad \textit{since } a_0 = 0 \\
&= a_{n+2}.
\end{aligned}$$

This completes the inductive step.

2. A box has m drawers; Drawer i contains g_i gold coins, s_i silver coins and ℓ_i lead coins, for $i = 1, 2, \ldots, m$. Assume that one drawer is selected randomly and that two randomly selected coins from that drawer turn out each to be of a distinct type. What is the probability that the chosen drawer is drawer 1?

Solution: Let B be the event that the coins are of a distinct type and let D_i be the event that drawer i is chosen. What we are asked for is

$$\Pr(D_1 \mid B) = \frac{\Pr(D_1 \cap B)}{\Pr(B)}.$$

Now

$$\Pr(D_i \cap B) = \frac{1}{m} \cdot \frac{g_i s_i + g_i \ell_i + s_i \ell_i}{(g_i + s_i + \ell_i)(g_i + s_i + \ell_i - 1)}$$

and so

$$\Pr(B) = \sum_{i=1}^{m} \Pr(D_i \cap B) = \frac{1}{m} \sum_{i=1}^{m} \frac{g_i s_i + g_i \ell_i + s_i \ell_i}{(g_i + s_i + \ell_i)(g_i + s_i + \ell_i - 1)}.$$

Therefore,

$$\Pr(D_1 \mid B) = \frac{g_1 s_1 + g_1 \ell_1 + s_1 \ell_1}{(g_1 + s_1 + \ell_1)(g_1 + s_1 + \ell_1 - 1)} \times \left(\sum_{i=1}^{m} \frac{g_i s_i + g_i \ell_i + s_i \ell_i}{(g_i + s_i + \ell_i)(g_i + s_i + \ell_i - 1)} \right)^{-1}.$$

3. A particle sits at the left hand end of a line $0 - 1 - 2 - \cdots - L$. When at 0 it moves to 1. When at $i \in [1, L-1]$ it makes a move to $i-1$ with probability $p \neq 1/2$ and a move to $i+1$ with probability $1-p$. When at L it stops.

   Let E_k denote the expected number of visits to 0 if we started the walk at k.

   (a) Explain why

$$\begin{aligned} E_L &= 0 \\ E_0 &= 1 + E_1 \\ E_k &= pE_{k-1} + (1-p)E_{k+1} \qquad \mathit{for}\ 0 < k < L. \end{aligned}$$

   (b) Given that $E_k = A\left(\frac{p}{1-p}\right)^k + B$ is a solution to your equations for some A, B, determine A, B and hence find E_0.

   Solution:

   $E_L = 0$ because the particle will not move from L. $E_0 = 1 + E_1$ because the particle must move to position 1 and then the expected number of extra visits to 0 will now be E_1. Finally,

   $E_k =$
   $\mathbf{E}(\#\ visits | moves\ right) \Pr(moves\ right) + \mathbf{E}(\#\ visits | moves\ left) \Pr(moves\ left)$
   $= pE_{k-1} + (1-p)E_{k+1}.$

   $E_L = 0$ implies then that $A\left(\frac{p}{1-p}\right)^L + B = 0$ and so $B = -A\left(\frac{p}{1-p}\right)^L$.
   $E_0 = 1 + E_1$ implies then that $A + B = 1 + \frac{p}{1-p}A + B$ which implies that $A = \frac{1-p}{1-2p}$.
   Thus

$$E_0 = \frac{1-p}{1-2p}\left(1 - \left(\frac{p}{1-p}\right)^L\right).$$