21-301 Combinatorics
Homework 4
Due: Monday, September 27

1. Suppose that you are asked to multiply a collection of $m \times m$ matrices to form the product $A_1 A_2 \cdots A_{n+1}$. Let $C_0 = 1$ and let C_n be the number of ways to do this. For example $C_2 = 2$. We can compute $(A_1A_2)A_3$ or $A_1(A_2A_3)$. Show that

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

   Determine C_n.

2. A box has m drawers; Drawer i contains g_i gold coins, s_i silver coins and ℓ_i lead coins, for $i = 1, 2, \ldots, m$. Assume that one drawer is selected randomly and that two randomly selected coins from that drawer turn out each to be of a distinct type. What is the probability that the chosen drawer is drawer 1?

3. A particle sits at the left hand end of a line $0 - 1 - 2 - \cdots - L$. When at 0 it moves to 1. When at $i \in [1, L-1]$ it makes a move to $i-1$ with probability $p \neq 1/2$ and a move to $i+1$ with probability $1-p$. When at L it stops.

   Let E_k denote the expected number of visits to 0 if we started the walk at k.

   (a) Explain why

$$\begin{aligned} E_L &= 0 \\ E_0 &= 1 + E_1 \\ E_k &= pE_{k-1} + (1-p)E_{k+1} \qquad for\ 0 < k < L. \end{aligned}$$

   (b) Given that $E_k = A\left(\frac{p}{1-p}\right)^k + B$ is a solution to your equations for some A, B, determine A, B and hence find E_0.