

21-301 Combinatorics  
 Homework 2  
 Due: Friday, September 18

1. Prove that for any  $k, n \geq 1$  that

$$\sum_{\substack{a_1 + \dots + a_{2k} = n \\ a_1, \dots, a_{2k} \geq 0}} 2^{a_1 + \dots + a_k} (-1)^{a_{k+1} + \dots + a_{2k}} \binom{n}{a_1, \dots, a_{2k}} = k^n.$$

**Solution:** We start with the expression

$$(x_1 + x_2 + \dots + x_{2k})^n = \sum_{\substack{a_1 + \dots + a_{2k} = n \\ a_1, \dots, a_{2k} \geq 0}} \binom{n}{a_1, \dots, a_{2k}} x_1^{a_1} x_2^{a_2} \dots x_{2k}^{a_{2k}}$$

and then put  $x_1 = x_2 = \dots = x_k = 2$  and  $x_{k+1} = x_{k+2} = \dots = x_{2k} = -1$  to get the result we want.

2. (a) Let  $\mathcal{S}_k$  denote the collection of  $k$ -sets  $\{1 \leq i_1 < i_2 < \dots < i_k \leq m - 4\} \subseteq [m]$  such that  $i_{t+1} - i_t \geq 5$  for  $1 \leq t < k$ . Show that

$$|\mathcal{S}_k| = \binom{m - 4k}{k}.$$

(b) How many of the  $5^n$  sequences  $x_1 x_2 \dots x_n$ ,  $x_i \in \{a, b, c, d, e\}$ ,  $i = 1, 2, \dots, n$  are there such that  $abcde$  does not appear as a consecutive subsequence e.g. if  $n = 6$  then we include  $adbcc$  in the count, but we exclude  $abcde$ .

[You should use Inclusion-Exclusion and expect to have your answer as a sum.]

**Solution:**

(a) For a first argument, let  $z_1 = i_1, z_2 = i_2 - i_1, \dots, z_k = i_k - i_{k-1}, z_{k+1} = m - i_k$ . We can count the number of choices for  $z_1, z_2, \dots, z_{k+1}$ . But these are the solutions to

$$z_1 + z_2 + \dots + z_{k+1} = m, \quad z_1 \geq 1, z_2, z_3, \dots, z_k \geq 5, z_{k+1} \geq 4.$$

The number of such is

$$\binom{m - 1 - 5(k - 1) - 4 + k + 1 - 1}{k + 1 - 1} = \binom{m - 4k}{k}.$$

Alternatively, we can represent a  $k$ -set by a sequence of  $k$  1's and  $m - k$  0's in the usual way. Now we need every pair of 1's separated by at least 4 0's. We can start with a sequence of  $m - 4k$  0's, choose  $k$  of them and replace each of these  $k$  0's by 10000. This process is reversible. For the 0,1 sequences we are counting each 1 is followed by at least 4 0's. Just replace 10000 by 0 to get a sequence of  $m - k$  0's.

(b) Let  $A = \{a, b, c, d, e\}^n$ . Then let

$$A_k = \{x \in A : x_k = a, x_{k+1} = b, x_{k+2} = c, x_{k+3} = d, x_{k+4} = e\}$$

for  $k = 1, 2, \dots, n - 4$ .

Let  $\mathcal{S} = \bigcup_{k \geq 0} \mathcal{S}_k$ . Then

$$|A_S| = \begin{cases} 5^{n-5|S|} & S \in \mathcal{S}_{|S|} \\ 0 & S \notin \mathcal{S}_{|S|} \end{cases}.$$

Then we must compute

$$\begin{aligned} \left| \bigcap_{i=1}^m \bar{A}_i \right| &= \sum_{S \in \mathcal{S}} (-1)^{|S|} |A_S| \\ &= \sum_{S \in \mathcal{S}_{|S|}} (-1)^{|S|} 5^{n-5|S|} \\ &= 5^n \sum_{k=0}^m (-1)^k |\mathcal{S}_k| 5^{-5k} \\ &= 5^n \sum_{k=0}^m (-1)^k \binom{m-4k}{k} 5^{-5k} \end{aligned}$$

3. How many ways are there of placing  $m$  distinguishable balls into  $n$  boxes so that no box contains more than  $m/3$  balls.

(You should use Inclusion-Exclusion and expect to have your answer as a sum.)

**Solution:** Let  $A$  be the set of  $n^m$  allocations of balls to boxes. Let  $A_i$  denote the allocations in  $A$  in which box  $i$  gets at least  $\mu = \lfloor m/3 \rfloor + 1$  balls. ( $\mu$  is the smallest integer greater than  $m/3$ ). We want the size of  $\bigcap_{i=1}^m \bar{A}_i$ .

Now

$$|A_i| = \sum_{k=\mu}^m \binom{m}{k} (n-1)^{m-k}$$

and

$$|A_{i,j}| = \sum_{k,l=\mu}^m \binom{m}{k,l,m-k-l} (n-1)^{m-k-l}$$

and  $A_S = \emptyset$  for  $|S| \geq 3$ . So,

$$\left| \bigcap_{i=1}^m \bar{A}_i \right| = n^m - n \sum_{k=\mu}^m \binom{m}{k} (n-1)^{m-k} + \binom{n}{2} \sum_{k,l=\mu}^m \binom{m}{k,l,m-k-l} (n-2)^{m-k-l}.$$