21-301 Combinatorics
Homework 2
Due: Wednesday, September 8

1. Prove that for any $k, n \geq 1$ that

$$\sum_{\substack{a_1+\cdots+a_{2k}=n \\ a_1,\ldots,a_{2k}\geq 0}} 2^{a_1+\cdots+a_k}(-1)^{a_{k+1}+\cdots+a_{2k}}\binom{n}{a_1,\ldots,a_{2k}} = k^n.$$

2. **(a)** Let $\mathcal{S}_k$ denote the collection of k-sets $\{1 \leq i_1 < i_2 < \cdots < i_k \leq m-4\} \subseteq [m]$ such that $i_{t+1} - i_t \geq 5$ for $1 \leq t < k$. Show that

$$|\mathcal{S}_k| = \binom{m-4k}{k}.$$

   (b) How many of the 5^n sequences $x_1x_2\cdots x_n$, $x_i \in \{a,b,c,d,e\}$, $i = 1,2,\ldots,n$ are there such that *abcde* does not appear as a consecutive subsequence e.g. if $n = 6$ then we include *adbbce* in the count, but we exclude *abcdea*.

   [You should use Inclusion-Exclusion and expect to have your answer as a sum.]

3. How many ways are there of placing m distinguishable balls into n boxes so that no box contains more than $m/3$ balls.
   (You should use Inclusion-Exclusion and expect to have your answer as a sum.)